Does Majority Rule Produce Hasty Decisions?*

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Abstract. In collective decision-making, impatient or low-stake voters are attractive allies if some group members want to stop deliberations to decide for their favored alternative. Competition for their votes can lead to hasty outcomes through strategic complementarity in the stopping decision. Majority rule is not robust in that one very impatient agent can cause the whole group to rush to a decision. This can be avoided by super-majority rule. Moreover the impacts of impatient voters and patient voters are not symmetric: one extremely patient agent will not cause the group to deliberate without end under super-majority rule unless unanimity is required.

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1. Introduction

From the Condorcet jury theorem to the more recent literature on strategic voting (e.g., Austen-Smith and Banks 1996; Feddersen and Pesendorfer 1997), much has been said about voting as a mechanism of aggregating private information to reach a common decision. Most collective decision processes, however, involve more than simply casting a vote. There is typically a deliberative stage when members of the group jointly grapple with the issues and try to explore the best option to take. Consider a standard-setting committee formed by a group of firms deciding on what technological platform to adopt. Engineers from the companies as well as outside experts will test and openly discuss the strengths and weaknesses of the proposed standards and perhaps come up with ways to improve them. Similarly, countries deciding on whether and how to cut greenhouse gas emissions will engage a panel of scientists to gather evidence to guide their actions. In jury decision-making, even though the evidence is already presented at trial, one can think of the jury deliberation stage as the process by which jurors together examine and learn from the evidence to form an opinion for a verdict.[1]

While a committee cannot wait forever until the evidence becomes unequivocal, the quality of collective decisions will suffer if they are reached hastily. A controversial issue in the climate change debate, for instance, is precisely about how much scientific ambiguity we should tolerate before taking concrete actions. How do group members with divergent interests reach a collective decision when they have the option to collect more information to shed light on the issues? Under what circumstances will decisions be made too hastily? Among other things, answers to these questions are relevant for the design of collective decision-making rules. When comparing majority rule to super-majority rule in *The Federalist No. 58*, James Madison acknowledged that the super-majority requirement "might have been an additional shield to some particular interests, and another obstacle generally to hasty and partial measures" (Hamilton et al. 1982, p. 298), though he favored majority rule on other grounds.

The trade-off between the cost of collecting additional information and the benefit from making a more informed decision is a classic problem in statistics. Abraham Wald (1947) was the first to study a model of sequential sampling in which an individual decision-maker has to choose between collecting more information (at a cost) and deciding in favor of one of two alternative hypotheses. The solution takes the form of a stopping rule and is characterized by two thresholds. If the accumulated evidence in terms of the likelihood ratio crosses an upper threshold, then one action should be taken. If the accumulated evidence crosses a lower threshold, then the alternative action should

[1] In their classic study of the American jury, for example, Kalven and Zeisel (1966) find that the length of jury deliberation is positively related to the length of the trial, indicating that learning from the evidence is a time-consuming process.

be taken. Otherwise, the decision-maker should keep searching for more information.[2]

In this paper we adapt Wald's framework to study a deliberation game where the stopping decision is made collectively. In our model, a group of agents must choose between two alternatives. Members of the group have different preferences for the alternatives as well as different discount rates. At each moment each member of the group votes for one of the two alternatives or abstains. The game ends when a sufficient number of agents vote for one of the alternatives. Meanwhile, members continuously update their beliefs about the relative merit of the alternatives as new information arrives publicly. Just as in the single-agent case, each voter's strategy is characterized by two thresholds: he votes for one alternative if his belief reaches beyond an upper threshold, votes for the other alternative if his belief falls below a lower threshold, and abstains otherwise. However, the thresholds adopted by an agent will typically be different from what he would have chosen were he the sole decision-maker. In a collective decision the influence of each agent is constrained by the decisions of others. Since the optimal upper threshold for a person depends on where the lower threshold is, strategic interactions in the stopping decisions are important. As is standard in voting games, our model contains a plethora of Nash equilibria where no single individual has any influence on the outcome. We find that multiple equilibria may exist besides these trivial equilibria. Thus, expectation can play an important role in the deliberation process. Agents may decide slowly if they expect others are taking their time, but they may also rush to a decision to preempt others from prematurely reaching a different decision.

A particularly striking result is that, under majority rule, mutual expectations of quick decisions may cause an alternative to be chosen almost instantaneously in equilibrium, as long as there is at least one agent in the group who is extremely impatient. We show that this near collapse of deliberation can be avoided by a super-majority rule. Moreover, the impacts of impatient voters and patient voters on the equilibrium amount of group deliberations are not symmetric: the presence of one extremely patient agent will not cause the group to deliberate without end under super-majority rule (unless unanimity is required). In this sense, super-majority rule can be an effective shield against hasty decisions. If voters can choose the decision-making rule, there are environments in which super-majority rule is preferred to simple majority rule by a majority of voters.

Agents may rush to a decision because the strategic effects of waiting are non-monotone. Specifically, if an agent thinks that one of the alternatives would be adopted by the group with too little evidence (relative to his unconstrained optimal threshold), his best response is to vote to adopt the other alternative with insufficient evidence as well. Furthermore, in this case, accelerating the adoption of one alternative would induce him to vote to adopt the other alternative earlier: the two stopping decisions are strategic complements. But

[2] Wald and Wolfowitz (1948) showed that this "sequential probability ratio test" is optimal in the sense that it requires on average the least observations among all tests with the same power. Wald's original contribution has been extended in numerous directions. See, for example, Siegmund (1985).

if the agent thinks that one alternative would be adopted with excessive deliberations, his best response is again to vote to adopt the other alternative *earlier* (relative to his unconstrained optimal threshold). Furthermore, in this case, delaying the adoption of one alternative causes the agent to accelerate his adoption of the other alternative: the two stopping decisions are strategic substitutes. Intuitively, whenever the threshold for the adoption of one alternative deviates away from its unconstrained optimal value in either direction, the payoff from waiting falls relative to the immediate payoff from taking the other alternative. Therefore, the agent responds by stopping earlier for the other alternative.

This non-monotonicity means that, for each agent, his best-response upper threshold is always below the unconstrained optimal value. Similarly, his best-response lower threshold is always above the unconstrained optimal value. In other words, individuals respond to a loss of control over the adoption of one alternative by cutting short the deliberation before adopting the other alternative. Because of heavy discounting, a very impatient voter is willing to stop to take one alternative (say, α) at almost any state of evidence. Another voter, who is less impatient, may be prepared to wait for more information before deciding in favor of β if he were the sole decision-maker. But when he faces the impatient voter, this voter will want to stop immediately for his favored alternative β lest the evidence swings the other way and α is adopted prematurely by the impatient voter if the impatient voter is pivotal. Our paper shows that, under majority rule, an extremely impatient voter is indeed pivotal for one of the alternatives, so that the strategic complementarity that we describe leads to an almost complete collapse of group deliberations. Under super-majority rule, on the other hand, extremely impatient voters will be pivotal only when there is a sufficiently large number of them. Therefore, a suitably chosen super-majority rule can avoid the collapse of group deliberations when the group contains a few very impatient members. Moreover, super-majority rules are robust to the presence of a few extremely patient members in the group. Since a very patient agent does not want to stop until the evidence is very strong one way or the other, he is seldom pivotal in the stopping decision unless the super-majority rule requires almost unanimous agreement.

Our paper does not claim that super-majority rule is always better than majority rule. Indeed, when there is heterogeneity only in the preference dimension or only in the discount rate dimension, majority rule is preferred to super-majority rule by a majority of voters. With heterogeneity in both dimensions, however, super-majority rule can be a more robust decision-making mechanism in the presence of a few impatient voters, because it can prevent the possibility of a collapse of deliberations. Even in equilibria where deliberation does not collapse, the presence of impatient voters in the group may still produce too little deliberation under majority rule. A super-majority rule errs on the other side. By requiring the consent of more agents before reaching a decision, it tends to produce too much deliberation. However we use a stylized example to illustrate that too much deliberation under super-majority rule can be a lesser evil than insufficient deliberation under majority rule. In the example, even though neither majority rule nor super-majority rule induces the first-best outcome, almost all the agents (except the impatient ones) prefer super-majority rule to majority rule when their belief is relatively neutral.

The acquisition of public information during the decision-making process is an important but relatively unexplored area of research. The bulk of the existing economics literature focuses on the incentives of committee members to acquire or reveal private information. Some papers (e.g., Coughlan 2000; Ottaviani and Sorensen 2001; Krishna and Morgan 2004; Austen-Smith and Feddersen 2006) model group deliberations as cheap talk that may potentially overcome the strategic information revelation problem. A different strand of the voting literature considers the free-riding problem that committee members face when they collect their individual pieces of information (e.g., Li 2001; Martinelli 2007; Koriyama and Szentes 2009). We depart from these papers in that information acquisition is modeled as a joint decision made by the committee and that the evidence so produced is public. Not only is this kind of public learning an important feature of many collective decision procedures, but it also allows us to abstract away from issues arising from private information and directly address the question of whether decisions are made with too much haste or with undue delay.[3]

Our paper is related to recent contributions that adopt a collective search approach (Albrecht, Anderson and Vroman 2010; Compte and Jehiel 2010; 2011; Moldovanu and Shi 2010).[4] In these papers, a new alternative appears every period, and the group's decision is whether to adopt that alternative or to continue with search. In our paper, the alternatives are fixed but new information about the existing alternatives appears over time, and the group's decision is when to stop collecting evidence and decide which alternative to adopt. In our model, when the two thresholds are controlled by two different agents, each agent will also become "less picky" and adopt an alternative earlier than he would when both thresholds were under his control. Similarly, Albrecht, Anderson and Vroman (2010) find that in a heterogeneous search committee, each committee member will apply a lower acceptance standard than what he would have were he the sole decision maker. However, because the search process does not yield any new information about existing alternatives, no agent will ever change his mind and support an alternative that he previously rejected, as an impatient agent may do in our model. As a result, in their model having one committee member who is willing to accept any proposal will not cause the rest of the committee to stop searching entirely.

[3]Damiano, Li and Suen (2010) study a model in which delay cost induces people to reveal their private information over time. There is no new information that arrives during this process, and in their model decisions which are made earlier in equilibrium are always better than decisions which are made late.

[4]Also related are the papers by Strulovici (2010) and Messner and Polborn (2008), which explore collective decision-making problems when individuals have the option to learn about their own preferences over time.

Still more closely related to our paper are Gul and Pesendorfer (forthcoming) and Lizzeri and Yariv (2011). Gul and Pesendorfer (forthcoming) studies the competition between two political parties to provide information that may influence voters' choice. Each party's decision is characterized by a stopping rule, and these stopping decisions are strategically linked. Our paper studies the strategic interactions of the stopping decisions, but in the context of group decision-making. We also emphasize the disproportionate influence that a few impatient agents may have on the entire group. Lizzeri and Yariv (2011) adopts Wald's framework to study committee deliberations.[5] Their emphasis is on the interaction between voting rules and deliberation rules, and they identify factors (e.g., greater preference heterogeneity, larger super-majority requirements) that lead to greater deliberations. Our focus is on the non-monotonic nature of the strategic interactions among voters. We provide necessary and sufficient conditions for equilibrium, and identify the possibility of non-trivial multiple equilibria. We also emphasize voter heterogeneity in more than one dimension, and show how majority rule may not be a robust decision-making mechanism in the presence of a few impatient voters.

2. Information Acquisition in Collective Decision-Making

A group of $2m-1$ agents are choosing between two alternatives, α and β. The payoff to each action depends on the underlying state $\omega \in \{A, B\}$. In state A, agent i's payoff from α is 1, and his payoff from β is 0. In state B, agent i's payoff from α is 0, and his payoff from β is e^{v_i}. Hence v_i denotes the intensity of agent i's preference for β relative to α. Without loss of generality, we assume that v_i increases weakly in i. The agents do not observe the state, but they share a common prior belief. It is convenient to represent the belief about the states by the log ratio of the probabilities of the two states. We let $\theta_0 = \log(\Pr[\omega = A]/\Pr[\omega = B])$ represent the initial belief. With this parameterization, the probability of state A is given by $e^{\theta}/(1+e^{\theta})$ when the belief is θ. The immediate expected payoff from choosing α is higher than that from choosing β for agent i if and only if $\theta \geq v_i$.

We model the group decision-making process in continuous time. At each time t, each agent independently votes for α or β, or abstains. We focus on the class of majoritarian decision rules. Under decision rule $k \in \{m, m+1, \ldots, 2m-1\}$, an alternative is adopted at time t if it is supported by k agents or more. The decision process (i.e., voting) continues if neither α nor β receives sufficient votes. The decision rule $k = m$ corresponds to majority rule; $k > m$ is a super-majority rule, with $k = 2m-1$ being the unanimity rule. Each agent i discounts the future at a rate r_i. If an alternative is chosen at time t, then agent i's payoff is discounted by the factor $e^{-r_i t}$. In general, agents are heterogeneous in

[5] Other papers that uses Wald's framework in the economics literature include Moscarini and Smith (2001), which studies how a single decision-maker should alter the level of experimentation; Brocas and Carrillo (2009), which analyzes the option value of risky alternatives; and Sobbrio (2010), which considers the incentive of news editors to collect news.

both the preference dimension v_i and the impatience dimension r_i.

We abstract away from the issue of information disclosure by assuming that information is publicly observed by the group. It is common in economics to model the cost of information acquisition through discounting. So we assume that information arrives exogenously and continuously as long as an alternative has not yet been chosen by the group. One interpretation is that the group continues to incorporate new information and viewpoints as members keep deliberating on the merits of the two alternatives.[6]

The arrival of information is represented by a Wiener process $\mathrm{d}S$ that has a positive drift μ and an instantaneous variance ρ^2 if the state is A, or drift $-\mu$ and instantaneous variance ρ^2 if the state is B. For any time $t > 0$, the accumulated evidence S_t is a sufficient statistic for all the information that has arrived before t. The log-likelihood ratio of observing $S_t = s$ under the two states is

$$\log \frac{h((s-\mu)/\rho)}{h((s+\mu)/\rho)} = \frac{2\mu s}{\rho^2},$$

where $h(\cdot)$ is the standard normal density function. Hence a higher observed value of the accumulated evidence S_t is stronger evidence in favor of state A. Bayes' rule says the log posterior probability ratio is equal to the sum of the log prior probability ratio and the log-likelihood ratio. So, if we let $S'_t = 2\mu S_t/\rho^2$, the common belief of the group at time t is given by:

$$\theta_t = \theta_0 + S'_t.$$

Denote $\mu' \equiv 2\mu^2/\rho^2$. The re-defined process $\mathrm{d}S'$ is a Wiener process with drift μ' and instantaneous variance $2\mu'$ under state A, or drift $-\mu'$ and instantaneous variance $2\mu'$ under state B.[7] A higher value of μ' (higher μ or lower ρ) indicates a more informative deliberative process.

The crucial assumption here is that information is publicly observed and every agent has the same model of the world (i.e., uses the same likelihood function). As a result, every agent updates his belief in the same way during the information acquisition process. The assumption of common prior is not important. If agent i has prior belief $\theta_0^i \neq \theta_0$, we can simply re-define his prior to θ_0 and his preference parameter to $v_i + \theta_0 - \theta_0^i$.

If agent i were to make the decision alone, this would be a standard problem in optimal

[6] In the final section of this paper, we discuss another possible formulation of this model in which agents have to pay to keep information coming, and provide an alternative interpretation of impatient agents as voters with low stakes in the decision concerned.

[7] The same equation for the belief evolution can be obtained through a Brownian motion approximation to a model with discrete time and binomial signals. Specifically, let the length of each period be $\mathrm{d}t$. Suppose there is a binary signal each period that indicates the state is A with probability $1/2 + \sqrt{\mu'\mathrm{d}t/8}$ when the true state is A, or with probability $1/2 - \sqrt{\mu'\mathrm{d}t/8}$ when the true state is B. Then the total number of excess signals for A at time t converges to our diffusion process S'_t as $\mathrm{d}t$ goes to zero.

stopping. The solution to this problem is well known: there exists two threshold beliefs, denoted g_i^* and G_i^*, such that agent i should choose α if $\theta_t \geq G_i^*$; he should choose β if $\theta_t \leq g_i^*$; and he should wait for more information if $\theta_t \in (g_i^*, G_i^*)$. In the context of collective decision-making, in principle an agent's decision at time t could be a function of the whole sample path of S_t, his own decisions, and other agents' decisions prior to t. However, since for any agent i the difference between the expected payoffs of α and β depends solely on the current belief θ_t and is strictly increasing in it, we focus exclusively on equilibria in which agents adopt Markov cutoff strategies. Formally, let

$$\Sigma \equiv \left\{ (g, G) \in \Re^2 \mid g \leq G \right\}$$

denote the set of all cutoff strategies. For $g_i < G_i$, strategy $\sigma_i = (g_i, G_i)$ means voting for α when $\theta_t \geq G_i$, voting for β when $\theta_t \leq g_i$, and abstaining when $\theta_t \in (g_i, G_i)$. For $g_i = G_i$, strategy (g_i, G_i) means voting for α when $\theta_t \geq G_i$ and voting for β when $\theta_t < g_i$.

Figure 1 depicts the agents' cutoffs in a group of three individuals with decision rule $k = m = 2$. In this figure, α is adopted when θ_t reaches G_1, and β is adopted when θ_t reaches g_2. More generally, for any strategy profile $\sigma = (\sigma_1, \ldots, \sigma_{2m-1})$, let

$$G^{[k]}(\sigma) = \min \{\theta \mid \#\{i | \theta \geq G_i\} \geq k\}$$

denote the smallest belief for which at least k agents vote for alternative α, and let

$$g^{[k]}(\sigma) = \max \{\theta \mid \#\{i | \theta \leq g_i\} \geq k\}$$

denote the largest belief for which at least k agents vote for alternative β. Under decision rule k, if $G^{[k]}(\sigma) > g^{[k]}(\sigma)$, then α will be adopted by the group when $\theta_t \geq G^{[k]}(\sigma)$ and β will be adopted when $\theta_t \leq g^{[k]}(\sigma)$. If $G^{[k]}(\sigma) = g^{[k]}(\sigma)$, then α is adopted when $\theta \geq G^{[k]}(\sigma)$ and β is adopted when $\theta_t < g^{[k]}(\sigma)$.

Let $u_i(g, G \mid \theta)$ represent the payoff to agent i when the belief is θ and the stopping boundaries for α and β are G and g, respectively. Let the random variable T denote the first time the updated belief θ_t hits g or G given that the current belief is θ. We have:

$$u_i(g, G \mid \theta) = \begin{cases} \frac{e^\theta}{1+e^\theta} & \text{if } \theta \geq G, \\ \frac{e^\theta}{1+e^\theta} \Psi_i(g, G \mid \theta; A) + \frac{e^{v_i}}{1+e^\theta} \psi_i(g, G \mid \theta; B) & \text{if } \theta \in (g, G), \\ \frac{e^{v_i}}{1+e^\theta} & \text{if } \theta \leq g; \end{cases}$$

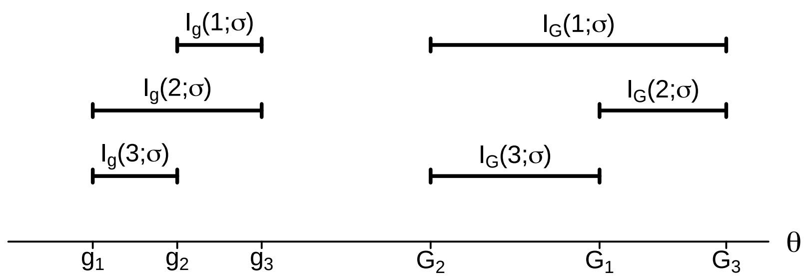

Figure 1. *Cutoffs and spans of control*

where, for $\omega \in \{A, B\}$,

$$\Psi_i(g, G \mid \theta; \omega) \equiv \mathrm{E}\left[e^{-r_i T} \mid \theta_T = G, \theta_0 = \theta; \omega\right] \Pr\left[\theta_T = G \mid \theta_0 = \theta; \omega\right], \tag{1}$$

$$\psi_i(g, G \mid \theta; \omega) \equiv \mathrm{E}\left[e^{-r_i \theta T} \mid \theta_T = g, \theta_0 = \theta; \omega\right] \Pr\left[\theta_T = g \mid \theta_0 = \theta; \omega\right]. \tag{2}$$

When $\theta \in (g, G)$, agent i's payoff would depend on which threshold is reached first and on the length of time it takes to reach it. In (1), $\Pr[\theta_T = G \mid \theta_0 = \theta; \omega]$ is the probability that θ_t reaches G before g in state ω, and $\mathrm{E}[e^{-r_i T} \mid \theta_T = G, \theta_0 = \theta; \omega]$ is the expected discount factor conditional on state ω and on θ_t reaching G before g. Explicit forms for the Ψ_i and ψ_i functions can be obtained from the analysis of first passage time in the theory of Brownian motion.[8] These formula are standard in the literature (Stokey 2009, Proposition 5.3; see also Cox and Miller 1965, pp. 210–213):

$$\Psi_i(g, G \mid \theta; A) = \frac{e^{-R_1(\theta - g)} - e^{-R_2(\theta - g)}}{e^{-R_1(G-g)} - e^{-R_2(G-g)}}, \tag{3}$$

$$\psi_i(g, G \mid \theta; B) = \frac{e^{-R_1(G-\theta)} - e^{-R_2(G-\theta)}}{e^{-R_1(G-g)} - e^{-R_2(G-g)}}; \tag{4}$$

with

$$R_1 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{4r_i}{\mu'}}\right), \quad R_2 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4r_i}{\mu'}}\right). \tag{5}$$

Note that $R_2 > 1$, $R_1 < 0$, and $R_1 + R_2 = 1$. In general the values of R_1 and R_2 depend on

[8]Equivalently, the functions Ψ_i and ψ_i can be obtained by stochastic calculus. For θ in the waiting region (g, G), the payoff function satisfies $u_i(g, G \mid \theta) = e^{-r_i \mathrm{d}t}\mathrm{E}\left[u_i(g, G \mid \theta + \mathrm{d}S')\right]$. Since $\mathrm{d}S'$ is a diffusion process with expected drift $\mu'(e^\theta - 1)/(1 + e^\theta)$ and instantaneous variance $2\mu'$, we can use Ito's lemma to derive a differential equation in u_i. Solving this differential equation and imposing the value-matching condition at the boundaries of the waiting region will give the explicit solutions for Ψ_i and ψ_i.

i, but we omit this dependence in the notation to avoid clutter.

The following lemma describes the key properties of u_i. All proofs are provided in the Appendix.

Lemma 1. *The following properties hold for any agent i and any threshold pair (g, G) with $g < G$.*

1. *Dynamic consistency: For $\theta \in (g, G)$,*
   (a) $\partial u_i / \partial G|_{(g,G;\theta)} > (=)\ 0$ if and only if $\partial u_i / \partial G|_{(g,G;\theta=G)} > (=)\ 0$;
   (b) $\partial u_i / \partial g|_{(g,G;\theta)} > (=)\ 0$ if and only if $\partial u_i / \partial g|_{(g,G;\theta=g)} > (=)\ 0$.
2. *Single crossing:*
   (a) For $G' > G$, $\partial u_i / \partial G|_{(g,G';\theta=G')} \geq 0$ implies $\partial u_i / \partial G|_{(g,G;\theta=G)} > 0$;
   (b) for $g' < g$, $\partial u_i / \partial g|_{(g,G;\theta=g)} \geq 0$ implies $\partial u_i / \partial g|_{(g',G;\theta=g')} > 0$.
3. *Minimal deliberation: For small $\epsilon > 0$, $u_i(v_i - \epsilon, v_i + \epsilon \mid v_i) > e^{v_i}/(1 + e^{v_i})$.*

The partial derivative of u_i with respect to either threshold reflects the trade-off between the cost of delay and the value of additional information. Part 1 of Lemma 1 says that this trade-off is dynamically consistent. If an agent prefers changing a threshold when he reaches it, then he will prefer the same change before reaching it. This means that we can solve the optimal stopping of an agent without reference to his current belief. Part 2 says that u_i is single-peaked in a threshold when the other is hold constant. Intuitively, the marginal waiting cost is constant but the value of additional information decreases as the agent is more sure about the state. If an agent prefers waiting to taking α at θ, he certainly prefers waiting to taking α at a smaller θ', where the case for α is weaker and there is more to gain from waiting. Finally, at belief $\theta = v_i$ agent i is indifferent between α and β. Part 3 of the lemma says that agent i prefers a small waiting region symmetric about v_i to immediate action. This follows from the assumption that θ_t follows a continuous diffusion process.

In the classic individual optimal-stopping problem, an agent i chooses (g_i, G_i) to maximize $u_i(g, G \mid \theta)$. The group decision model differs in that the influence of each agent on the final outcome is constrained by the cutoffs of other agents. Let $I_G(i;\sigma)$ and $I_g(i;\sigma)$ denote agent i's "spans of control" for the upper and lower thresholds, respectively. Formally, we have:

$$I_G(i;\sigma) \equiv \begin{cases} [G^{[k]}(\sigma), G^{[k+1]}(\sigma)] & \text{if } G_i < G^{[k]}(\sigma), \\ [G^{[k-1]}(\sigma), G^{[k+1]}(\sigma)] & \text{if } G_i = G^{[k]}(\sigma), \\ [G^{[k-1]}(\sigma), G^{[k]}(\sigma)] & \text{if } G_i > G^{[k]}(\sigma); \end{cases}$$

$$I_g(i;\sigma) \equiv \begin{cases} [g^{[k]}(\sigma), g^{[k+1]}(\sigma)] & \text{if } g_i < g^{[k]}(\sigma), \\ [g^{[k-1]}(\sigma), g^{[k+1]}(\sigma)] & \text{if } g_i = g^{[k]}(\sigma), \\ [g^{[k-1]}(\sigma), g^{[k]}(\sigma)] & \text{if } g_i > g^{[k]}(\sigma). \end{cases}$$

Figure 1 illustrates. Under majority rule, since neither agent 2 nor 3 votes for α when $\theta < G_2$, α can never be adopted before G_2; and since both agents 2 and 3 vote for α when $\theta \geq G_3$, α must be adopted by G_3. As a result, agent 1 can only control at what beliefs between G_2 and G_3 that α will be adopted. Therefore, $I_G(1;\sigma) = [G_2, G_3]$. Similarly, agent 1 can only control at what beliefs between g_2 and g_3 that β will be adopted. Choosing $g_1 < g_2$ effectively means β will be adopted by the group when the belief reaches g_2. Therefore, his span of control over the adoption of β does not extend beyond $I_g(1;\sigma) = [g_2, g_3]$.

Definition 1. *Let $(\hat{g}, \hat{G}) \equiv (g^{[k]}(\sigma), G^{[k]}(\sigma))$. A strategy profile σ is an equilibrium under decision rule k if the following conditions are satisfied for any agent i:*

1. *For any belief θ and any $(g', G') \in I_g(i;\sigma) \times I_G(i;\sigma)$, $u_i(\hat{g}, \hat{G} \mid \theta) \geq u_i(g', G' \mid \theta)$.*
2. *(a) $G_i > (<)\ \hat{G}$ if $\partial u_i / \partial G|_{(\hat{g},\hat{G};\theta=\hat{G})} > (<)\ 0$;*
   (b) $g_i > (<)\ \hat{g}$ if $\partial u_i / \partial g|_{(\hat{g},\hat{G};\theta=\hat{g})} > (<)\ 0$.
3. *$(g_i, G_i) \neq (v_i, v_i)$.*

Condition 1 of Definition 1 is the standard Nash equilibrium requirement. Note that when $k \neq 2m-1$, for any pair of thresholds (g, G), it would be a Nash equilibrium in our model for all agents to adopt the same strategy (g, G) because each agent's span of control for each alternative would be a singleton. To rule out this type of equilibria, we require that each agent's strategy be sensitive to his own preference at the margin. Condition 2 says that agent i does not vote for an alternative at the threshold if his marginal gain from extending the threshold is positive, and, conversely, that he votes for an alternative before the belief reaches the threshold for that alternative if his marginal gain from contracting the threshold is positive. Finally, condition 3 requires that no agent i will immediately vote for an alternative whenever he prefers that alternative to the other at his current belief. By part 3 of Lemma 1 we know that voting for the preferred alternative immediately is never optimal for an agent.[9]

3. Strategic Best Response in Stopping Decisions

It is useful to first study a one-sided stopping problem in which each agent takes one of the thresholds as given and determines the optimal value of the other threshold. It is straightforward to use Lemma 1 to establish that the solution to a one-sided stopping problem is finite and unique. For any agent i, define $\phi_i(G)$ as the lower best-response function for agent i which gives, for any fixed G, the optimal lower cutoff g that maximizes $u_i(g, G \mid \theta)$ subject to $g \leq G$. Similarly, the upper best-response function $\Phi_i(g)$ is defined as the optimal upper cutoff G that maximizes $u_i(g, G \mid \theta)$ subject to $G \geq g$, given

[9] Without condition 3 it would be an equilibrium for all agent i to adopt $(g_i, G_i) = (v_i, v_i)$ when $v_1 = \ldots = v_{2m-1}$, even though the strategy is not optimal to any agent. Such a strategy does not violate condition 2 because (v_i, v_i) is a saddle point for individual agent i.

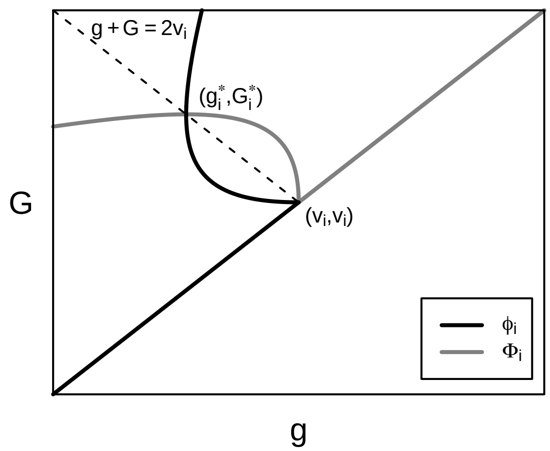

Figure 2. *A typical pair of best-response functions*

any fixed g.

Figure 2 depicts a typical pair of ϕ_i and Φ_i. The optimal thresholds for agent i in a single-agent two-sided stopping problem are denoted by g_i^* and G_i^*. Obviously, if agent i can choose both thresholds without constraint, the solution (g_i^*, G_i^*) is characterized by a fixed point of (ϕ_i, Φ_i).[10] The center of the unconstrained optimal waiting region, $(G_i^* + g_i^*)/2$, only depends on preference v_i; while its width, $G_i^* - g_i^*$, only depends on discount rate r_i. Notice in Figure 2 that there is another fixed point (v_i, v_i), but it is not optimal due to the minimal deliberation result of Lemma 1.

Since ϕ_i and Φ_i are symmetric across the line $g + G = 2v_i$, we focus on the properties of ϕ_i in the following discussion.[11] An important feature of ϕ_i is that it coincides with the 45-degree line when $G \leq v_i$. Recall that agent i prefers the immediate adoption of β to the immediate adoption of α when the belief is lower than v_i. If the upper threshold for adopting α is set below v_i, agent i would still prefer β to α when his belief reaches the upper threshold. Hence, the optimal decision for agent i is to adopt β immediately. So $\phi_i(G) = G$. Note that this property is true for any r_i no matter how small. Even an extremely patient agent i will choose to adopt β immediately when $G \leq v_i$.

[10]The values of the optimal thresholds can be explicitly solved. In particular, we have $G_i^* = v_i + \log(-R_2/R_1)/(R_2 - R_1)$ and $g_i^* = v_i - \log(-R_2/R_1)/(R_2 - R_1)$.

[11]Formal statements and proofs of these properties are provided in Lemma 2 in the Appendix.

By contrast, when $G > v_i$, even an extremely impatient agent i will prefer waiting to adopting β immediately when his belief is above g but sufficiently close to it. The optimal lower threshold in this case is characterized by first-order condition:

$$\left.\frac{\partial u_i}{\partial g}\right|_{(\phi_i(G),G;\theta=\phi_i(G))} = 0.$$

Note that $\phi_i(G)$ is continuous at v_i in Figure 2. Suppose the upper boundary is fixed at a level just above v_i. Since the difference in expected payoff between α and β is very small when the belief reaches this boundary, there is very little gain for agent i to wait for more information before deciding to settle for alternative β. As a result, the best response is to set the lower threshold at a level just below v_i.

Another important feature of ϕ_i is that it is non-monotone for $G \geq v_i$. In Figure 2, $\phi_i(G)$ decreasse as G increases from $G = v_i$, reaches a minimum as G approaches G_i^*, and increase as G continue to increase past G_i^*. This means that for all $G \geq v_i$, the one-sided best response, $\phi_i(G)$, is greater than $\phi_i(G_i^*) = g_i^*$, the optimal lower threshold when agent i controls both thresholds. Intuitively, G_i^* is the optimal upper threshold for agent i. When $G < G_i^*$, α is chosen "too early." When $G > G_i^*$, it is chosen "too late." In either case the payoff from waiting for agent i would be lower than what it would be if $G = G_i^*$. The result that a loss of control (in this case, from controlling both boundaries to controlling one boundary) may reduce the incentive to obtain information is discussed in Albrecht, Anderson and Vroman (2010) in the context of a collective search model and by Strulovici (2010) in a two-armed bandit experiment setting.

The non-monotonicity of ϕ_i and Φ_i implies there is a strategic difference between too much waiting and too little waiting. Imagine that the two thresholds are controlled by two different agents. If the agent who controls the upper threshold decides to wait too long before accepting α (from the perspective of the agent who controls the upper threshold), then the value of waiting falls and the agent who controls the lower threshold responds by adopting β earlier: excessive deliberation is self-limiting. But if the agent who controls the upper threshold decides to wait too little before accepting α, the value of waiting also falls and the other agent again cuts short his waiting before choosing β. In this case, rushing to a decision can become self-reinforcing. This property raises the possibility of multiple equilibria, in which expectations about other agents' stopping decisions can play an important role in the deliberation process.

Figure 3 illustrates how ϕ_i varies with r_i. As agent i becomes more impatient, he raises the lower threshold for the adoption of β. As this agent becomes extremely impatient, he sets a lower threshold arbitrarily close to the upper threshold: the best-response function is very close to the 45-degree line. Note that in this case $\phi_i(G) > v_i$ if G is very high, so that at the stopping point, β is actually worse than α for agent i. But since he cannot influence the stopping decision for α, he would prefer to stop for the inferior alternative

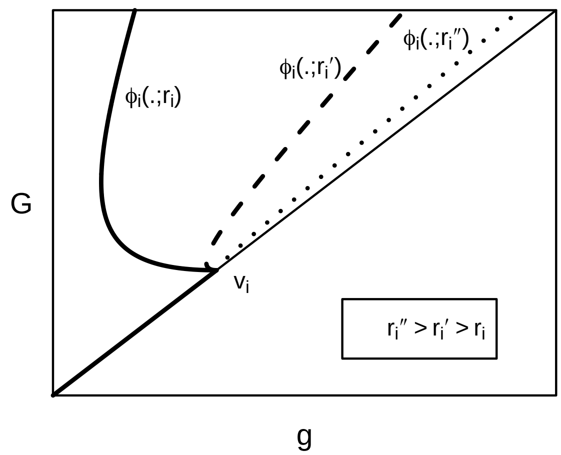

Figure 3. *The best-response curve bends toward the 45-degree line as an agent becomes more impatient*

than to bear the cost of deliberations.

4. Equilibrium Analysis

Given decision rule k, define the "pivotal" best-response functions:

$$\Phi_{piv}(g;k) \equiv \min\{\theta \mid \#\{i|\Phi_i(g) \leq \theta\} \geq k\}, \tag{6}$$

$$\phi_{piv}(G;k) \equiv \max\{\theta \mid \#\{i|\phi_i(G) \geq \theta\} \geq k\}. \tag{7}$$

These two functions are, respectively, the k-th smallest upper best-response for a given g and the k-th largest lower best-response for a given G. In the following we sometimes suppress the argument k when doing so does not cause confusion.

In Figure 4, panel (a), we plot the lower best-response functions of three agents, with $v_1 < v_2 < v_3$ and $r_1 > r_2 = r_3$. If the upper boundary is fixed at $G < G'$, agent 2 wants to stop to take alternative β before agent 3 but after agent 1. Hence, under majority rule, $\phi_{piv}(G)$ coincides with $\phi_2(G)$. However because agent 1 is more impatient than the other agents, his lower best-response function ϕ_1 cuts ϕ_2 and ϕ_3. For $G \in [G', G'')$, $\phi_1(G)$ is between $\phi_2(G)$ and $\phi_3(G)$. Hence $\phi_{piv}(G) = \phi_1(G)$ in this range. For $G \geq G''$, the pivotal lower best-response function coincides with $\phi_3(G)$. In panel (b), the upper best-response functions of the three agents do not intersect. Hence the pivotal best-response function $\Phi_{piv}(g)$ is simply the best-response function of agent 2.

Because agents' preferences over a single threshold are single-peaked (part 2 of Lemma 1), it is a weakly dominant strategy for each agent to choose his best response in a one-

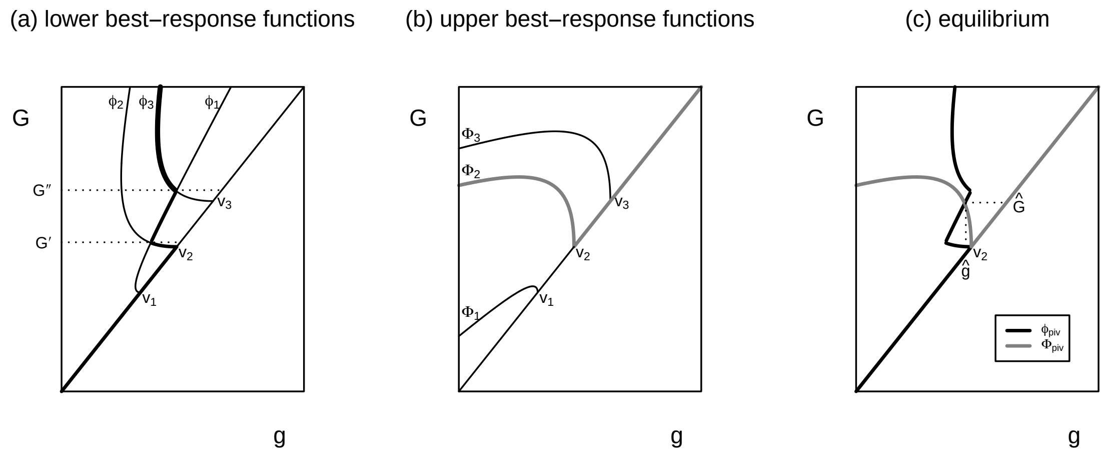

Figure 4. *Pivotal best-response functions and equilibrium*

sided optimal-stopping problem where one of the thresholds is fixed. Thus, it is natural to expect that in a one-sided optimal-stopping problem the group decision would be $\phi_{piv}(G;k)$ (when the upper threshold is fixed at G) or $\Phi_{piv}(g;k)$ (when the lower threshold is fixed at g).

The analysis is more complicated when the agents are choosing both thresholds simultaneously, because there is no weakly dominant strategy in the two-sided problem. Nevertheless, we can show that so long as the agents' equilibrium behavior satisfies condition 2 of Definition 1, the equilibrium outcome would be as if for each threshold each agent is choosing a best response against the equilibrium value of the other threshold.

By the single-crossing property of Lemma 1, we can restate condition 2(a) of Definition 1 as requiring that, for each agent i, $G_i > (<) \hat{G}$ if $\Phi_i(\hat{g}) > (<) \hat{G}$. Suppose σ is an equilibrium of the game. Any agent j with $\Phi_j(g^{[k]}(\sigma)) < G^{[k]}(\sigma)$ must adopt a upper threshold $G_j < G^{[k]}(\sigma)$, while any agent j with $\Phi_j(g^{[k]}(\sigma)) > G^{[k]}(\sigma)$ must adopt a upper threshold $G_j > G^{[k]}(\sigma)$. It follows that any agent i with $G_i = G^{[k]}(\sigma)$ must have $\Phi_i(g^{[k]}(\sigma)) = G^{[k]}(\sigma)$. Furthermore, since $G^{[k]}(\sigma)$ is the k-th smallest upper threshold among all agents, the number of agent j with $\Phi_j(g^{[k]}(\sigma)) < G^{[k]}(\sigma)$ must be no greater than $k-1$, while the number of agents with $\Phi_j(g^{[k]}(\sigma)) > G^{[k]}(\sigma)$ must be no greater than $2m-k-1$. Hence, $G^{[k]}(\sigma) = \Phi_{piv}(g^{[k]}(\sigma);k)$. By the same logic, $g^{[k]}(\sigma) = \phi_{piv}(G^{[k]}(\sigma);k)$. Thus, in any equilibrium the agent who "controls" a threshold (in the sense that he can both speed up and delay the adoption of an alternative) must be choosing his one-sided best response, and, furthermore, his one-sided best response must coincide with the pivotal best response as defined in (7).[12]

[12]Note that in general non-pivotal agents needs not choose their one-sided best responses and, hence, may not stop in the same order as their best responses.

Define

$$\hat{\Sigma}(k) \equiv \left\{ (\hat{g}, \hat{G}) \in \Sigma \mid \hat{g} = \phi_{piv}(\hat{G}; k),\ \hat{G} = \Phi_{piv}(\hat{g}; k),\ \hat{g} < \hat{G} \right\}$$

as the set of fixed points of (ϕ_{piv}, Φ_{piv}) such that the upper threshold is strictly higher than the lower threshold. The argument in the last paragraph implies that an equilibrium outcome must belong to $\hat{\Sigma}(k)$. The following proposition shows that the converse is also true.

Proposition 1. *For any majoritarian rule $k \in \{m, m+1, \ldots, 2m-1\}$, an equilibrium of the deliberation game exists. Furthermore, a strategy profile σ is an equilibrium under decision rule k if and only if:*

1. *$(g^{[k]}(\sigma), G^{[k]}(\sigma)) \equiv (\hat{g}, \hat{G}) \in \hat{\Sigma}(k)$.*
2. *For any agent i,*
   (a) $G_i > (<)\ \hat{G}$ if $\Phi_i(\hat{g}) > (<)\ \hat{G}$;
   (b) $g_i > (<)\ \hat{g}$ if $\phi_i(\hat{G}) > (<)\ \hat{g}$.
3. *For any agent i, $(g_i, G_i) \neq (v_i, v_i)$.*

Conditions 2 and 3 of Proposition 1 are restatements of conditions 2 and 3 of Definition 1. To prove Proposition 1, we need to show that if σ satisfies conditions 1–3 of Proposition 1, then $(g^{[k]}(\sigma), G^{[k]}(\sigma))$ is a solution to the constrained optimization problem for each agent i and for every θ:

$$\max_{g,G} u_i(g, G \mid \theta) \quad \text{subject to} \quad (g, G) \in I_g(i; \sigma) \times I_G(i; \sigma). \tag{8}$$

The proof proceeds by first showing that $(g^{[k]}(\sigma), G^{[k]}(\sigma))$ satisfies the Kuhn-Tucker necessary conditions for agent i's constrained optimization problem (8). We then show that any threshold pair that satisfies the Kuhn-Tucker necessary conditions must be a solution to the constrained optimization problem. This step is not immediate because the objective function u_i is not quasi-concave. Finally, equilibrium existence is proved through Lemma 3 in the Appendix, which establishes that $\hat{\Sigma}(k)$ is non-empty. Panel (c) of Figure 4 illustrates that the equilibrium thresholds are given by the intersection of the pivotal best-response functions.

Proposition 2. *There exists a unique pair of equilibrium thresholds when all agents have the same v_i or the same r_i. When agents differ in both v_i and r_i, multiple equilibrium outcomes may exist. If $(\hat{g}, \hat{G})$ and $(\hat{g}', \hat{G}')$ are both equilibrium waiting regions, then either $(\hat{g}, \hat{G}) \supset (\hat{g}', \hat{G}')$ or $(\hat{g}', \hat{G}') \supset (\hat{g}, \hat{G})$. Finally, under majority rule $k = m$, any equilibrium waiting region must contain v_m in its interior.*

When all agents have the same v_i or the same r_i, their best-response functions do not cross. If all r_i's are the same, Φ_{piv} is equal to the upper best-response function of the agent

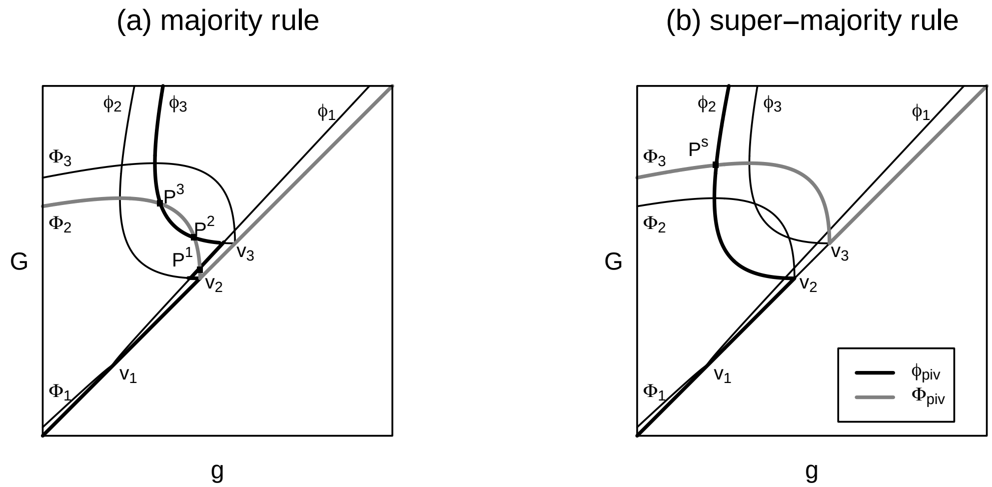

Figure 5. *Equilibria under different decision rules*

with k-th smallest v_i, while ϕ_{piv} is equal to lower best-response function of the agent with the k-th largest v_i. If all v_i's are the same, Φ_{piv} and ϕ_{piv} are equal to the upper and lower best-response functions of the agent with the k-th largest r_i. In both cases, $\hat{\Sigma}(k)$ would be a singleton.

Proposition 2 also says that there may be multiple equilibrium outcomes when agents differ in both v_i and r_i. Figure 5 depicts a three-agent example with $v_1 < v_2 < v_3$ and $r_1 > r_2 = r_3$. Note that ϕ_1 cuts across ϕ_2 and ϕ_3. Because agent 1 is less patient than agents 2 and 3, despite his weaker preference for alternative β he is willing to adopt β at an earlier point than agents 2 and 3 are when G is large. Suppose the decision rule is majority rule (i.e., $k = 2$). In panel (a), the pivotal lower best-response function, $\phi_{piv}(\cdot, 2)$, is represented by the think line. It intersects $\Phi_{piv}(\cdot, 2)$ (which in this example is the same as Φ_2) four times. The three intersections other than (v_2, v_2), denoted by P^1, P^2, and P^3 in the figure, represent the equilibrium outcomes of this game under majority rule.

When the model exhibits multiple equilibria, Proposition 2 says that the equilibrium waiting regions are nested and, in the case of majority rule, must contain v_m. There is thus a "most patient" equilibrium outcome and a "least patient" equilibrium outcome. In panel (a) of Figure 5, P^3 is the most patient and P^1 is the least patient, and P^1, P^2, P^3 all contain v_m. Multiple equilibrium outcomes exist because the two stopping thresholds are strategic complements up to a certain point. In general there is no natural way the agents can coordinate on a particular equilibrium. Different agents may have different preferences over different equilibria. More importantly, even the same agent may have different preferences over two equilibria as belief changes. Hence, while a majority may prefer one equilibrium outcome at some belief, another majority may prefer another one at a different belief.

If the upper and lower thresholds are controlled by the same agent i in equilibrium of this game, then the equilibrium thresholds are simply (g_i^*, G_i^*). In general, however, the upper and lower thresholds may be controlled by different agents in equilibrium. Let $G^*_{(j)}$ denote the j-th largest value among $\{G_1^*, \ldots, G_{2m-1}^*\}$, and let $g^*_{(j)}$ denote the j-th smallest value among $\{g_1^*, \ldots, g_{2m-1}^*\}$. The following proposition says that the equilibrium waiting region under decision rule k is narrower than that implied by what would obtain if each agent i acted naively by using g_i^* and G_i^* as their thresholds.

Proposition 3. *If $(\hat{g}, \hat{G})$ is an equilibrium under decision rule k, then $\hat{g} \geq g^*_{(k)}$ and $\hat{G} \leq G^*_{(k)}$.*

Intuitively, strategic interactions induce each agent to become "less picky," in the sense that he does not demand as much evidence before adopting an alternative when he loses control of the adoption of the other alternative to other agents. Albrecht, Anderson and Vroman (2010) obtain a similar result in the context of collective search.

Few general comparative statics results can be obtained for the equilibrium thresholds $\hat{g}$ and $\hat{G}$. However, since $G - \phi_{piv}(G)$ and $\Phi_{piv}(g) - g$ are monotone, we can derive comparative statics results for the equilibrium width of the waiting region. Let $\overline{y}(\mathbf{v}, \mathbf{r}, k)$ and $\underline{y}(\mathbf{v}, \mathbf{r}, k)$ denote the widest and narrowest equilibrium width of the waiting region under decision rule k when preferences are $\mathbf{v} = (v_1, \ldots, v_{2m-1})$ and discount rates are $\mathbf{r} = (r_1, \ldots, r_{2m-1})$.

Proposition 4. *Both $\overline{y}(\mathbf{v}, \mathbf{r}, k)$ and $\underline{y}(\mathbf{v}, \mathbf{r}, k)$ are decreasing in $\mathbf{r}$ and increasing in k.*

Proposition 4 says that having a group of more impatient agents or a smaller majority requirement shorten the deliberation process in the sense that the waiting regions of both the most patient and least patient equilibrium will become narrower. It is important to note that Proposition 4 does not apply to all equilibria. For example, in Figure 5, panel (a), the waiting region of P^2 will become wider when agent 3 becomes less patient. It is straightforward to show that in general the waiting period of an equilibrium decreases in $\mathbf{r}$ if and only if $(\mathrm{d}\phi_{piv}/\mathrm{d}G)(\mathrm{d}\Phi_{piv}/\mathrm{d}g) < 1$ (i.e., ϕ_{piv} cross Φ_{piv} from below) at the equilibrium.

Lizzeri and Yariv (2011) show that when all agents have the same deliberation costs and when preferences are symmetric (i.e., $v_j + v_{2m-j} = 2v_m$), the lower threshold decreases in k while upper threshold increases in k in equilibrium. Our result that $\overline{y}$ and $\underline{y}$ increases with the size of the required majority k does not place any restriction on the profile of preferences and discount rates, but we cannot derive unambiguous results for individual thresholds. Lizzeri and Yariv (2011) also show that in their symmetric environment, greater preference heterogeneity (i.e., a decrease in v_j with a corresponding increase in v_{2m-j} for $j < m$) leads to more deliberations. This result does not hold in general when agents have heterogeneous discount rates.

5. Rushing to a Decision

In our model a change in preference or discount rate of one agent not only affects his own voting decisions, but other members' voting decisions as well through his influence on their spans of control and on the determination of who "controls" the thresholds in equilibrium. Unlike in Compte and Jehiel (2010) and Lizzeri and Yariv (2011), the determination of the identity of the "key members" who controls the stopping decisions is complex in our model because of two-dimensional heterogeneity in both preferences and discount rates. In this section we show how strategic complementarity in the stopping decision gives impatient agents a disproportionate influence on collective deliberation, because the value of deliberation to the patient agents falls as impatient agents try to adopt a decision quickly without waiting for much information. A striking result is that, for a group of any size, the presence of merely one very impatient agent can cause the whole group to rush to a decision under majority rule.

Figure 6 illustrates the effect on equilibrium deliberation as one of the agents becomes extremely impatient. In panel (a), if $r_1 = r_2 = r_3$, the best-response function ϕ_1 of agent 1 lies to the left of ϕ_2 and ϕ_3. Under majority rule agent 1 is never pivotal, and equilibrium is given by the intersection of ϕ_2 and Φ_2, i.e., point P^a. As r_1 increases, the best-response function ϕ_1 bends toward the 45-degree line and cuts ϕ_2. This means that agent 1 is willing to adopt decision β before agent 2 does even though agent 1 has a lower preference for β. In panel (b) of Figure 6, we see that agent 1 becomes pivotal in the sense that $\phi_{piv}(G)$ coincides with $\phi_1(G)$ for some range of values of G. Equilibrium is given by the intersection of ϕ_1 and Φ_2. Note that agent 2 responds to agent 1 by stopping earlier to adopt α, because he expects the impatient agent would stop early for decision β. The waiting region corresponding to equilibrium P^b is narrower compared to that in equilibrium P^a (P^b is closer to the 45-degree line than P^a is). As r_1 increases further, ϕ_1 cuts Φ_2 at a point closer and closer to the 45-degree line. Panel (c) shows that the equilibrium P^c gets closer and closer to the point (v_2, v_2), which means that the waiting region becomes arbitrarily short. The whole group rushes to a decision even though only one of its members is impatient. In this sense majority rule is not a robust decision-making mechanism because the outcome can be dominated by the presence of just one very impatient agent.

The same result holds regardless of the size of the group and their preferences and discount rates. This general result is mainly driven by two properties of the best-response functions. First, for any agent j, $\Phi_j(g) = g$ for $g \geq v_j$ and $\Phi_j(g)$ is continuous at $g = v_j$. Thus, for g slightly below v_m, $\Phi_m(g)$ is below the upper thresholds of agents $m+1$ through $2m-1$ and is above the upper thresholds of agents 1 through $m-1$. In other words, agent m is pivotal in the adoption of α. Second, as r_i grows without bound, $G -$

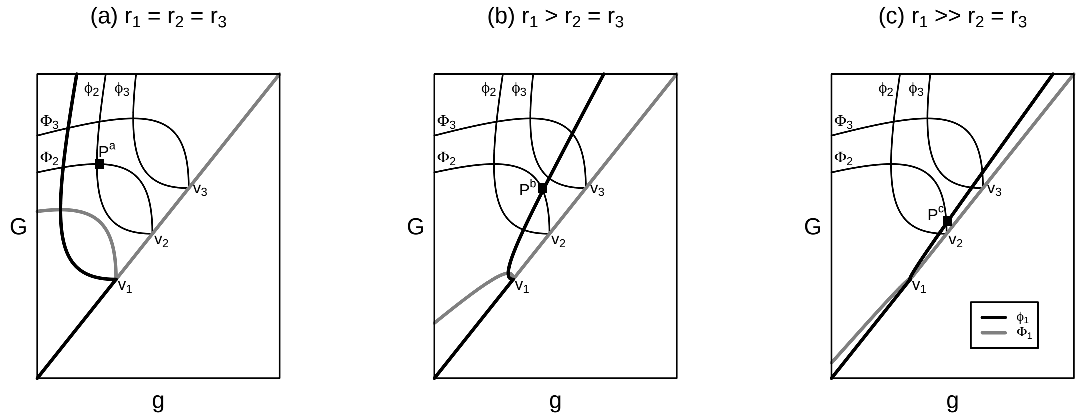

Figure 6. *Effect on equilibrium deliberation as agent* 1 *becomes very impatient*

$\phi_i(G)$ shrinks to 0 for any $G > v_i$. See Figure 3 above; the formal statement is given in Lemma 2 in the Appendix. Suppose, for example, that $i < m$. As r_i becomes very large, the curve ϕ_i cuts ϕ_j near the 45-degree line for all $j > i$. Thus, for G slightly above v_m, $\phi_i(G)$ is to the right of $\phi_m(G)$, meaning that, despite his lower preference for β, agent i is prepared to adopt β before agent m does. In this way agent i becomes pivotal in the adoption of β. Equilibrium is given by the intersection of Φ_m and ϕ_i. In this equilibrium the waiting region is very narrow and is centered near v_m. Agent i is prepared to wait very little because he is extremely impatient. Agent m also waits very little because the value of information is low as he is almost indifferent between the two alternatives inside the waiting region.

Proposition 5. *Let $v_1 < \ldots < v_{2m-1}$ and the decision rule be majority rule ($k = m$). For any $\epsilon > 0$, there exists $\bar{r}(\epsilon)$ such that if $r_i \geq \bar{r}(\epsilon)$ for some i, then there is an equilibrium $(\hat{g}, \hat{G})$ that is contained in the interval $[v_m - \epsilon, v_m + \epsilon]$.*

The collapse of deliberations described in Proposition 5 is a stark manifestation of the potential influence that a few impatient agents can bring to the collective decision-making process. One way to understand this to is think about these impatient agents as "swing voters." An impatient agent may be biased in favor of α, but because of discounting he is willing to settle for β as long as the evidence (i.e., the value of S'_t) swings a bit in favor of β. His vote can easily switch between adopting α and adopting β depending on small random changes in the information arrival process. As a result, swing voters are easy targets of alliance if members in the rest of the group want to adopt one of the two alternatives. To avoid some members of the group from capturing these swing voters to adopt one alternative, members who are biased for the other alternative will cut short the deliberative process by pushing forward the stopping threshold for their favored alternative as well. The effects of such behavior are magnified in equilibrium through strategic complementarity.

We may call the type of equilibrium in Proposition 5 a *hasty equilibrium*, because, starting at any initial belief $\theta_0 \in (\hat{g}, \hat{G})$, the time it takes for the belief $\theta_t = \theta_0 + S'_t$ to reach the boundaries of the waiting region is short as ϵ can be very small. In this equilibrium, which of the two alternatives will be chosen essentially depends only on whether the initial belief θ_0 is greater or less than the preference v_m of agent m. Even if θ_0 happens to lie within the waiting region $(\hat{g}, \hat{G})$, the probability of adopting the two alternatives is still largely independent of the true state. To see this, note from equation (1) that $\Pr[\theta_T = \hat{G} \mid \theta_0; A]$ is equal to $\Psi_i(\hat{g}, \hat{G} \mid \theta_0)$ evaluated at $r_i = 0$. From (3) and (5), we obtain:

$$\Pr[\theta_T = \hat{G} \mid \theta_0; A] = \frac{e^{\hat{G}} - e^{\hat{G}+\hat{g}-\theta_0}}{e^{\hat{G}} - e^{\hat{g}}}.$$

Similarly,

$$\Pr[\theta_T = \hat{G} \mid \theta_0; B] = \frac{e^{\theta_0} - e^{\hat{g}}}{e^{\hat{G}} - e^{\hat{g}}}.$$

Because the exponential function is convex, the probability of reaching decision α is higher in state A than in state B when $\theta_0 \in (\hat{g}, \hat{G})$. However, as $\hat{G} - \hat{g}$ shrinks to 0 while keeping $(\theta_0 - \hat{g})/(\hat{G} - \hat{g})$ fixed along the sequence, we have

$$\lim_{\hat{G}-\hat{g}\to 0} \Pr[\theta_T = \hat{G} \mid \theta_0; A] = \lim_{\hat{G}-\hat{g}\to 0} \Pr[\theta_T = \hat{G} \mid \theta_0; B] = \frac{\theta_0 - \hat{g}}{\hat{G} - \hat{g}}.$$

In other words, whether the true state is A or B has little effect on the probability of choosing α. In a hasty equilibrium, potential evidence about the state contained in the process S'_t plays almost no role in informing the group's decision.

It should be emphasized that the impacts of very impatient agents and very patient agents are not symmetric. While the presence of merely one extremely impatient agent can lead to a rush to making decisions under majority rule, the presence of a few extremely patient agent generally will not produce an equilibrium with never-ending deliberations. Intuitively, different group members compete for the support of very impatient agents to obtain their favored alternatives, resulting in a pre-mature stopping of deliberations which is magnified through strategic complementarity. However, it is difficult to court the support of very patient agents because they prefer to stop deliberating only when the evidence is unequivocal. Thus very patient agents are seldom pivotal in group decision-making. Moreover the non-monotonicity of the best-response functions means that the stopping decisions exhibit strategic substitutability when there is excessive waiting. Thus the presence of very patient agents will cause other agents to stop deliberating earlier, putting an upper bound on the equilibrium width of the waiting region (unless very patient agents comprise more than half of the group).

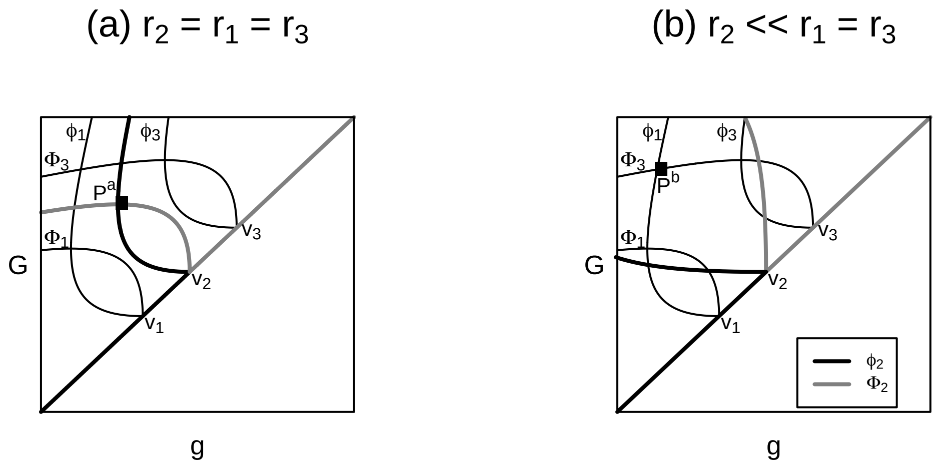

Figure 7. *Effect on equilibrium deliberation as agent 2 becomes very patient*

Proposition 6. *Let* $v_1 < \ldots < v_{2m-1}$ *and the decision rule be majority rule* ($k = m$)*. For any* $\zeta > 0$*, there exist* $\underline{r}(\zeta)$ *such that if* $\#\{i \mid r_i < \underline{r}(\zeta)\} \leq m - 1$*, then there is no equilibrium* $(\hat{g}, \hat{G})$ *such that* $\hat{G} - \hat{g} > \zeta$*.*

We can illustrate Proposition 6 with Figure 7. In panel (a), $r_1 = r_2 = r_3$. Agent 2 is pivotal for the adoption of both alternatives, and the equilibrium is indicated by P^a. As agent 2 becomes more patient, his upper best-response curve Φ_2 becomes more vertical while his lower best-response curve ϕ_2 becomes more horizontal. For low enough r_2, Φ_3 eventually cuts Φ_2, and ϕ_1 eventually overtakes ϕ_2. In panel (b), the equilibrium under majority rule is given by the intersection of Φ_3 and ϕ_1 (i.e., point P^b). Since agent 2 ceases to be pivotal, the equilibrium waiting region would not become any wider even if r_2 keeps decreasing to 0. A very patient agent demands very strong evidence before stopping to make a decision. It is difficult for other group members to court his vote in support of adopting the alternatives. Thus a very patient agent is unlikely to be pivotal, and his presence in the group does not trigger a cumulative response through strategic complementarity as an impatient agent does.

6. Super-majority Rule as an Obstacle to Hasty Measures

To avoid making hasty decisions, a majority of agents may choose to adopt a super-majority rule at the beginning of the deliberation process. From Proposition 4, the width of the waiting region in the most patient equilibrium and in the least patient equilibrium are increasing in the size of the majority requirement k. It is not surprising that people

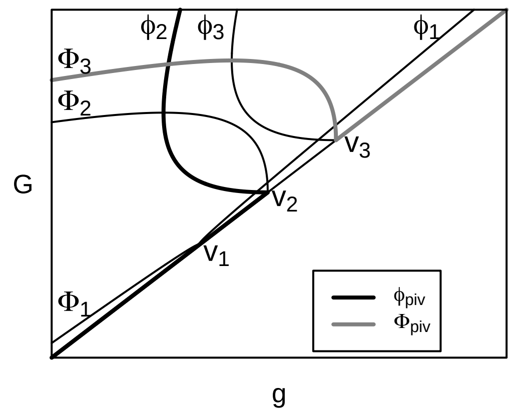

__Figure 8.__ Increasing the super-majority requirement removes the hasty equilibrium

generally deliberate longer when they adopt a decision rule with a greater super-majority requirement. More interestingly, a super-majority rule is more robust to the possibility that the presence of a few extremely impatient agents may cause the whole group to rush to a decision under majority rule.

We illustrate this point with Figure 8. The parameters used in this figure are reproduced from those in panel (c) of Figure 6. We re-interpret "agent 1" as a small group of n swing voters whose discount rate r_1 is very high. "Agent 2" and "agent 3" are interpreted as two large "factions" in the group, each of size $(2m-n-1)/2$. We assume that these two factions are large in the sense that a coalition between them will be decisive under rule k, i.e., $2m-n-1 \geq k$.

Suppose $k > n + (2m-n-1)/2$; that is, $n \leq 2(k-m)$. Then "agent 1" (the impatient swing voters) and "agent 3" (the faction with a high preference for β) together are still not sufficient to stop deliberations to adopt β without the consent of "agent 2." Thus the pivotal best-response function is $\phi_{piv} = \phi_2$. Likewise, the group has to secure the consent of "agent 3" for the adoption of α, which means $\Phi_{piv} = \Phi_3$. The equilibrium as shown in Figure 8 is not a hasty equilibrium. In other words a suitably designed super-majority rule $k > m$ can avoid the collapse of deliberations triggered by very impatient agents, unless the number of these very impatient agents exceeds $2(k-m)$. This conclusion can be generalized beyond the example shown in Figure 8. For any profile of preferences and discount rates, we have the following result.

Proposition 7. *Let $v_1 < \ldots < v_{2m-1}$. For any $\epsilon > 0$, there exists $\tilde{r}(\epsilon)$ such that if*

$$\#\{i \mid r_i \geq \tilde{r}(\epsilon)\} \leq 2(k - m), \tag{9}$$

then any equilibrium $(\hat{g}, \hat{G})$ under decision rule k must have $\hat{G} - \hat{g} \geq \epsilon$.

Note that majority rule $k = m$ is special in the sense that the mere presence of one very impatient agent violates condition (9). Proposition 7 implies that increasing the size of the supermajority by one will allow the number of very impatient agents to increase by two without causing deliberations to collapse to a hasty equilibrium. In fact, as long as extremely impatient agents comprise less that half of the group, Proposition 7 implies that a three-quarters super-majority requirement ($k \geq m + (m-1)/2$) is sufficient to rule out a hasty equilibrium.

However raising the super-majority requirement may also create a risk that some very patient agents could drag the whole group into protracted deliberations. In the Appendix, we prove a generalized version of Proposition 6 for any decision rule k. This generalized proposition states that as long as the number of very patient agents does not exceed $2m - k - 1$, there is an upper bound on the width of the equilibrium waiting region under decision rule k. Note that the maximum number of very patient agents that can be allowed without causing endless deliberations falls by one (instead of two) as the super-majority requirement is increased by one. For example, suppose that the number of very impatient agents and the number of very patient agents are both equal to $(m-1)/2$, and the remaining agents with non-extreme discount rates are in majority. Then Proposition 7 and the generalized version of Proposition 6 imply that a decision rule with between 5/8 and 3/4 super-majority requirement can rule out a hasty equilibrium without creating the risk of protracted deliberations.

A hasty equilibrium with an almost complete collapse of deliberations is an extreme manifestation of the strategic problem created by the competition for the support of impatient voters. Even when competition takes a less extreme form, strategic complementarity in the stopping decision can cause decisions to be made too quickly. We illustrate this point using the multiple equilibria example of Figure 5 discussed earlier.

In Figure 5, equilibrium P^1 corresponds to a hasty equilibrium if r_1 is very high. Two other equilibria, P^2 and P^3, exist. In these two equilibria competition for the support of the impatient agent 1 to form a majority causes the remaining patient agents to rush to a decision because they expect other patient agents to rush to their favored alternative. Even though agent 1 "controls" neither threshold in equilibrium P^2 or P^3, it is clear from the figure that the waiting region at either equilibrium is narrower than the width of the unconstrained optimal waiting region if member $j \neq 1$ were to make the decision alone (the latter is indicated by, say, the intersection between Φ_3 and ϕ_3 in Figure 5). Note also that the premature termination of deliberations in equilibrium P^2 or P^3 does not rely on agent 1 being extremely impatient. A moderately high r_1 is sufficient to produce this type of equilibria.

Suppose the decision rule in this example is changed to requiring super-majority, i.e., $k > m$. Then adopting either alternative would require the support of both agent 2 and agent 3. The relevant pivotal functions become ϕ_2 and Φ_3, with a unique equilibrium at point P^s, as shown in panel (b) of Figure 5. While equilibria P^1, P^2, and P^3 under majority rule induce too little deliberation relative to the unconstrained optimum (for group members other than agent 1), equilibrium P^s under super-majority rule induces too much deliberation. The choice between majority rule and super-majority rule in this example therefore boils down to comparing which of these inefficiencies is larger. Because individuals have different preferences and discount rates, a general welfare comparison of the voting rules does not yield any particular insights. In this paper our main focus is the disproportionate influence of a very small number of impatient agents on equilibrium deliberation. Therefore we will disregard the welfare of agent 1 and consider only the equilibrium payoffs to agents 2 and 3. If we interpret "agent 1" as impatient swing voters, we are in effect assuming that these voters are the minority and restricting our attention to the welfare of the remaining mainstream voters.

Proposition 8. *There is an interval containing $(v_2 + v_3)/2$ such that if the initial belief θ_0 belongs to that interval, then every agent (except agent 1) prefers equilibrium P^s under super-majority rule to equilibria P^1, P^2, and P^3 under majority rule. Furthermore there is no initial belief such that every agent (except agent 1) prefers majority rule to super-majority rule.*

The welfare comparison of equilibrium outcomes depends on the initial belief. It is clear that P^s cannot be preferred to, say, P^3 by all agents at any belief. For example, when the belief is near the upper boundary of the waiting region $(\hat{g}^s, \hat{G}^s)$ under P^s, agent 2 prefers adopting α immediately (which is the outcome in equilibrium P^3 under majority rule) to waiting. Similarly, when the belief is near the lower boundary $\hat{g}^s$, agent 3 prefers adopting β immediately to waiting. Nevertheless, Proposition 8 establishes that there is a range of beliefs near the center of the waiting region $(\hat{g}^s, \hat{G}^s)$ for which both agent 2 and agent 3 prefer super-majority rule to any equilibrium under majority rule.

The crucial step in establishing Proposition 8 is the following comparison:

$$0 < (\hat{G}^s - \hat{g}^s) - y^* < y^* - (\hat{G}^3 - \hat{g}^3), \tag{10}$$

where $y^* = G_j^* - g_j^*$ is the width of the unconstrained optimal waiting region for agent $j = 2, 3$. This result (proved in the Appendix) says that while super-majority rule produces too much deliberation (relative to the unconstrained optimal level y^*) and majority rule produces too little deliberation, the degree of distortion is relatively smaller under super-majority rule. At P^s, both Φ_{piv} and ϕ_{piv} are upward sloping (see panel (b) of Figure 5), meaning that the stopping decisions at the upper and lower boundaries are strategic substitutes: agent 2 compensates for a G above G_2^* by choosing a g above g_2^* and agent 3 compensates for a g below g_3^* by choosing a G below G_3^*. This is the "moderation effect" identified by Lizzeri and Yariv (2011). Hence the excessive deliberation induced by super-majority rule is self-limiting. At equilibrium P^3, however, the pivotal best-response functions are downward sloping. A lower G chosen by agent 2 induces agent 3 to react by choosing a higher g, which in turn prompts agent 2 to choose a still lower G. The distortions are magnified by strategic complementarity. This can be called a "radicalization effect," and is the result of the competitive courting of the impatient swing voters' support under majority rule.

We can illustrate these results with a numerical example. In Figure 9, we set $v_2 = -0.3$, $v_3 = 0.3$, and choose r and μ' such that $R_1 = -0.2$ and $R_2 = 1.2$. The waiting region corresponding to equilibrium P^s under super-majority rule is $(-1.55, 1.55)$, while the waiting region in equilibrium P^3 under majority rule is $(-0.88, 0.88)$. For r_1 sufficiently high, equilibrium P^1 is a hasty equilibrium with thresholds $(v_A - \epsilon, v_A + \epsilon)$. Panel (a) plots the utility of agent 2, and panel (b) plots the utility of agent 3. We see that utility at equilibrium P^3 is always above utility at the hasty equilibrium P^1. Figure 9 illustrates the point that welfare comparison of equilibrium outcomes depends on the initial belief. For example, agent 2 prefers equilibrium P^s to P^3 when the belief is low, but his preference switches when the belief is high. Agent 3 exhibits the opposite kind of preference reversal. Nevertheless, for $\theta_0 \in [-0.24, 0.24]$, utility at equilibrium P^s under super-majority rule is higher than utility at equilibrium P^3 under majority rule for both agents 2 and 3. Moreover, although P^s does not dominate P^3 for all beliefs, the range of beliefs for which agent 2 prefers P^3 to P^s does not overlap with the range of beliefs for which agent 3 prefers P^3 to P^s. In other words, there is no initial belief for which both agents prefer majority rule to super-majority rule.

7. Heterogeneity in One Dimension

We emphasize that the potential problems of majority rule that we describe are the result of heterogeneity in two dimensions: preferences and discount rates. Diverging preferences induce agents to court those with relatively high discount rates to support their favored alternatives, inducing agents with opposing preferences to react strategically that culminates in a short-circuiting of the deliberation process. If agents differ only in the preference dimension or only in the impatience dimension, however, majority rule can be more appealing than super-majority rule in our model.

Consider first the case where $v_1 \leq \ldots \leq v_{2m-1}$ and $r_1 = \ldots = r_{2m-1} = r$. Since best-response functions are ordered by preferences, the pivotal voter for α is agent k and the pivotal voter for β is agent $2m - k$ under decision rule k. Under majority rule, agent m is pivotal for both alternatives, and the equilibrium thresholds are the same as his unconstrained optimal thresholds. Note also that the unconstrained optimal width of the

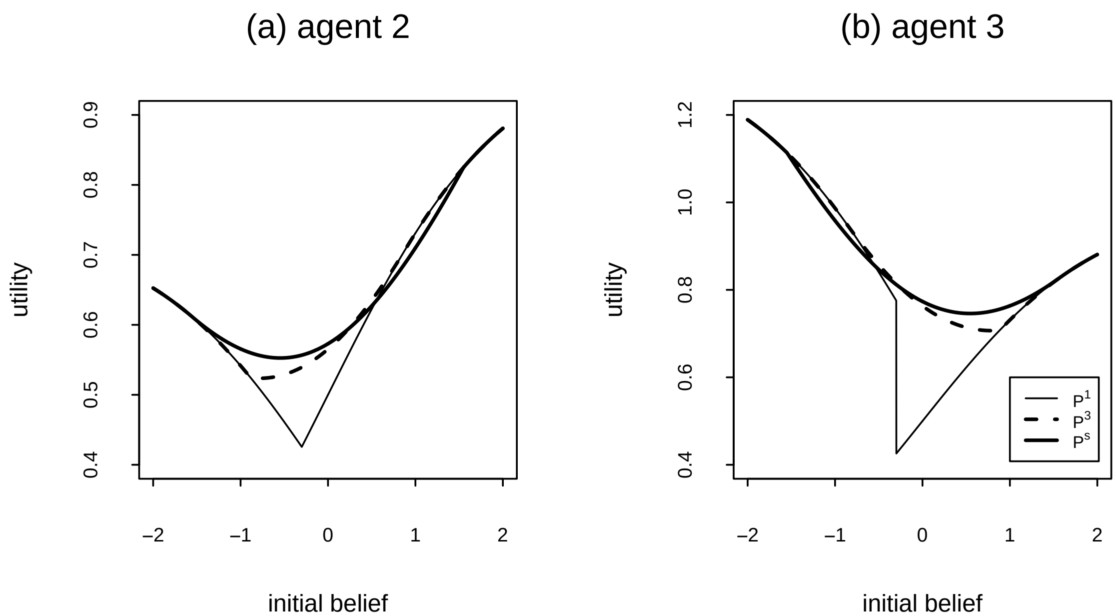

Figure 9. *Comparing utility for different agents in different equilibria*

waiting region y_m^* for agent m does not depend on v_m, so the equilibrium width y_m^* is equal to y_i^* for every agent i. Of course, the center of the equilibrium waiting region is v_m, which may not be ideal for other agents. Under super-majority rule $k > m$, equilibrium is determined by the intersection of the best-response functions Φ_k and ϕ_{2m-k}. The equilibrium width of the waiting region $\hat{y}$ is greater than y_m^*. We can establish the following result.

Proposition 9. *Suppose $r_1 = \ldots = r_{2m-1}$. For any initial belief θ_0, at least a majority of agents prefer majority rule to super-majority rule. Furthermore, there exists an interval of belief containing v_m such that, if θ_0 belongs to that interval, then all agents prefer majority rule to super-majority rule.*

Next consider the case $v_1 = \ldots = v_{2m-1} = v$. Without loss of generality, assume that $r_1 \geq \ldots \geq r_{2m-1}$. Since the best-response functions are ordered by discount rates, the pivotal best-response functions are Φ_k and ϕ_k under decision rule k. In other words, agent k is fully decisive, and the equilibrium waiting region is given by (g_k^*, G_k^*). A higher majority requirement causes an agent with a lower discount rate to become the decisive voter, so that the equilibrium waiting region expands. The following result again confirms that majority rule is desirable if there is heterogeneity in one dimension only.

Proposition 10. *Suppose $v_1 = \ldots = v_{2m-1}$. For any initial belief θ_0, at least a majority of agents (agents 1 to m) prefer majority rule to super-majority rule.*

8. Concluding Remarks

In this paper we provide a framework to study collective decision-making that takes times because of the need to accumulate evidence. We find that under majority rule impatient agents have a disproportionate influence on the decision-making process. The presence of merely one very impatient agent can cause the whole group to arrive at a decision with virtually no deliberation. In some cases, this hasty equilibrium is the unique equilibrium. But, as we have shown in Section 6, even when the hasty equilibrium is not unique, the presence of impatient agents may still trigger competition for their votes so that the amount of deliberation is reduced in all equilibria. As a result, a majority of agents may choose to adopt a super-majority rule to avoid hasty decisions.

While our setup uses discounting to model the cost of deliberation, a very similar model can be constructed with explicit deliberation costs, as in Wald's (1947) original formulation and in Lizzeri and Yariv (2011). Suppose, for agent i, the cost of information collection or deliberation is $\delta \mathrm{d}t$ for a time interval of length $\mathrm{d}t$, and the payoffs from α and β are λ_i and $\lambda_i e^{v_i}$, respectively. Agents with low values of λ_i have a relatively high cost of information collection. These low-stake agents do not care that much about which alternative is taken; their primary concern is to reduce the explicit information collection cost. In such an alternative setup, low-stake agents play a similar role as impatient agents in our model. If there is heterogeneity in both v_i and λ_i, competition for the votes of the low-stake agents can produce hasty decisions for the group under majority rule.

We assume that time is continuous so that an agent can change his vote anytime he wants. Modelling the deliberation process in discrete time would be tantamount to forcing the agents to wait a finite amount of time and observe a finite amount of information before they can vote again. This constraint will prevent an extremely impatient agent from swtiching his vote upon receiving an abiritraily small amount of information. We have not tried to solve out a discrete-time version of our model. We hypothesize that in such a model impatient agents would still tend to be disproportionately influential under majority rule for the reasons we have described, even though deliberation may not collapse almost entirely in the presence of one extremely impatient agent due to the minimum-waiting constraint. We expect that as the time period shrinks, this constraint will become unimportant, and the equlibrium outcome will converge to the continuous-time model.

While we argue that super-majority rule can be a shield against hasty decisions, we do not want to claim that it always dominates majority rule—Propositions 9 and 10 show that it is inferior when there is one-dimensional heterogeneity only. In a general environment with two-dimensional heterogeneity, it is difficult to say anything specific about the "optimal" decision rule even if we can agree on the appropriate social welfare function. What we have done in this paper instead is to identify a general tendency for impatient or low-stake agents to have a disproportionate impact on group deliberations, and suggest how super-majority rule is useful in guarding against such a tendency. While we have focused on the welfare of the decision makers, in many cases the decision also affects the welfare of people outside the decision-making group, who do not fully bear the deliberation cost. The deliberation cost for a jury, for example, is often insignificant compared to the effect of the verdict on the parties related to the case. In such cases, the robustness property may turn out to be more important than having rules which are "optimal" in a limited set of environments.

Appendix

A. Additional Lemmas

Lemma 2. *For each agent i, the best-response functions are given by:*

$$\Phi_i(g) = \begin{cases} \left\{ G \mid \partial u_i / \partial G|_{(g,G;\theta=G)} = 0 \right\} & \text{if } g < v_i, \\ g & \text{if } g \geq v_i; \end{cases}$$

$$\phi_i(G) = \begin{cases} \left\{ g \mid \partial u_i / \partial g|_{(g,G;\theta=g)} = 0 \right\} & \text{if } G > v_i, \\ G & \text{if } G \leq v_i. \end{cases}$$

The following properties hold for all i:

1. *Unconstrained optimum: The unconstrained optimal thresholds (g_i^*, G_i^*) is the unique fixed point of (ϕ_i, Φ_i) such that $(g_i^*, G_i^*) \neq (v_i, v_i)$. The optimal thresholds are symmetric, with $g_i^* + G_i^* = 2v_i$; and the optimal width $G_i^* - g_i^*$ depends only on r_i.*
2. *Differentiability: (a) $\Phi_i(g)$ is continuous everywhere and differentiable when $g \neq v_i$; (b) $\phi_i(G)$ is continuous everywhere and differentiable when $G \neq v_i$.*
3. *Waiting region: (a) $\Phi_i(g) - g$ is positive, strictly decreasing, and strictly concave for $g < v_i$; (b) $G - \phi_i(G)$ is positive, strictly increasing and strictly concave for $G > v_i$.*
4. *Non-monotonicity: (a) $\Phi_i(g)$ is increasing for $g < g_i^*$, and decreasing for $g \in [g_i^*, v_i)$; (b) $\phi_i(G)$ is decreasing for $G \in (v_i, G_i^*]$, and increasing for $G > G_i^*$.*
5. *Comparative Statics: (a) For any g, $\Phi_i(g)$ is increasing in v_i and decreasing in r_i; (b) for any G, $\phi_i(G)$ is increasing in v_i and increasing in r_i.*
6. *Effect of impatience: (a) For any $g_0 < v_i$ and $\epsilon > 0$, there exists $\bar{r}_i$ such that $\Phi_i(g) - g \leq \epsilon$ for any $g \in (g_0, v_i)$ if $r_i \geq \bar{r}_i$; (b) for any $G_0 > v_i$, $\epsilon > 0$, there exists $\bar{r}_i$ such that $G - \phi_i(G) \leq \epsilon$ for any $G \in (v_i, G_0)$ if $r_i \geq \bar{r}_i$.*

Lemma 3. *For any majoritarian decision rule $k \in \{m, \ldots, 2m-1\}$, the set $\hat{\Sigma}(k)$ is non-empty.*

Lemma 4. *For any agent i and any pair of thresholds (g, G) and (g', G') such that $\theta \in [g', G'] \subset [g, G]$,*

$$u_i(g, G \mid \theta) - u_i(g', G' \mid \theta) \begin{cases} > 0 & \text{if } \Phi_i(g) \geq G \text{ and } \phi_i(G) \leq g; \\ < 0 & \text{if } \Phi_i(g) \leq G' \text{ and } \phi_i(G) \geq g'. \end{cases}$$

B. Proofs

Proof of Lemma 1. *Part 1.* From the formulas for Ψ_i and ψ_i in (3) and (4), we have:

$$\frac{\partial \Psi_i}{\partial G} = \frac{-\left(R_2 e^{-R_2(G-g)} - R_1 e^{-R_1(G-g)}\right)\Psi_i}{e^{-R_1(G-g)} - e^{-R_2(G-g)}};$$
$$\frac{\partial \psi_i}{\partial G} = \frac{(R_2 - R_1)\, e^{-(G-\theta)}\Psi_i}{e^{-R_1(G-g)} - e^{-R_2(G-g)}}.$$

Therefore,

$$\frac{\partial u_i}{\partial G} = \frac{(R_2 - R_1)\, e^{-(G-g)} e^{\theta} \Psi_i}{(1+e^{\theta})\left(e^{-R_1(G-g)} - e^{-R_2(G-g)}\right)} \left(e^{v_i - g} - e^{f(r_i, G-g)}\right),$$

where

$$f(r_i, G-g) = \log \frac{R_2 e^{R_1(G-g)} - R_1 e^{R_2(G-g)}}{R_2 - R_1}.$$

For $\theta > g$, the first term is positive. Therefore, the sign of $\partial u_i / \partial G$ depends only on the sign of $v_i - g - f(r_i, G-g)$ and not on the value of θ. Similarly,

$$\frac{\partial u_i}{\partial g} = \frac{(R_2 - R_1)\, e^{-(G-g)} e^{v_i} \psi_i}{(1+e^{\theta})\left(e^{-R_1(G-g)} - e^{-R_2(G-g)}\right)} \left(e^{f(r_i, G-g)} - e^{G - v_i}\right).$$

For $\theta < G$, the first term is positive. Therefore, the sign of $\partial u_i / \partial g$ depends only on the sign of $f(r_i, G-g) - G + v_i$ and not on the value of θ.

Part 2. From the proof of part 1, $\partial u_i / \partial G|_{(g,G;\theta=G)}$ has the same sign as $v_i - g - f(r_i, G-g)$, which is strictly decreasing in G. Similarly, $\partial u_i / \partial g|_{(g,G;\theta=g)}$ has the same sign as $f(r_i, G-g) - G + v_i$, which is strictly decreasing in g.

Part 3. From the proof of part 1, $\partial u_i / \partial G|_{(v_i-\epsilon, v_i+\epsilon;\theta=v_i)}$ has the same sign as $\epsilon - f(r_i, 2\epsilon)$. Since this expression is equal to 0 and its derivative is equal to 1 at $\epsilon = 0$, we have $\partial u_i / \partial G|_{(v_i-\epsilon, v_i+\epsilon;\theta=v_i)} > 0$ for small $\epsilon > 0$. It follows that

$$u_i(v_i - \epsilon, v_i + \epsilon \mid v_i) > u_i(v_i - \epsilon, v_i \mid v_i) = e^{v_i}/(1 + e^{v_i}).$$ ■

Proof of Lemma 2. From the proof of Lemma 1, we know that the sign of $\partial u_i / \partial G$ depends only on the sign of $v_i - g - f(r_i, G-g)$ when $\theta \in [g, G]$. As $f(r_i, G-g) > 0$ when $G > g$, $v_i - g - f(r_i, G-g) \leq 0$ for any $g \geq v_i$. This shows that $\Phi_i(g) = g$ when $g \geq v_i$. Suppose now $g < v_i$. As $f(r_i, 0) = 0$, $\partial u_i / \partial G > 0$ at $G = g$. As $\lim_{G\to\infty} f(r_i, G-g) = \infty$, $\partial u_i / \partial G < 0$ when G is sufficiently large. It follows that there must exist some $G' > g$ such that $\partial u_i / \partial G = 0$ at $G = G'$. By Lemma 1, G' must maximize $u_i(g, G \mid \theta)$ with respect to G, subject to the constraint that $G \geq g$ for all $\theta \in (g, \Phi_i(g)]$. Finally, for all $\theta > \Phi_i(g)$, $u_i(g, \Phi_i(g) \mid \theta) = e^\theta/(1+e^\theta) \geq u_i(g, G \mid \theta)$ for all $G \geq g$. The argument for $\phi_i(G)$ is analogous.

Part 1. For $g < v_i < G$ the first-order conditions for the optimal thresholds characterize the best-response functions:

$$v_i - g - f(r_i, \Phi_i(g) - g) = 0,$$
$$f(r_i, G - \phi_i(G)) - G + v_i = 0,$$

where $f(r_i, y) = \log\left((R_2 e^{R_1 y} - R_1 e^{R_2 y})/(R_2 - R_1)\right)$. Adding these two equations shows that Φ_i and ϕ_i are symmetric in the sense that $\Phi_i(g) + \phi_i(G) = 2v_i$ for $g + G = 2v_i$. By the minimal deliberation result of Lemma 1, we know that for small $\epsilon > 0$,

$$\Phi_i(v_i - \epsilon) \geq v_i + \epsilon \quad \text{and} \quad \phi_i(v_i + \epsilon) \leq v_i - \epsilon.$$

Furthermore, from the first-order conditions, we know that for large η,

$$\Phi_i(v_i - \eta) < v_i \quad \text{and} \quad \phi_i(v_i + \eta) > v_i.$$

It then follows from the symmetry of ϕ_i and Φ_i that (ϕ_i, Φ_i) must have a fixed point that is not (v_i, v_i) on the line $g + G = 2v_i$. Part 3 of this Lemma establishes that both $\Phi_i(g) - g$ and $G - \phi_i(G)$ are increasing and concave. Hence (ϕ_i, Φ_i) has only one fixed point with $G_i^* > g_i^*$. Finally, notice that the first-order conditions imply

$$-(G_i^* - g_i^*) + 2f(r_i, G_i^* - g_i^*) = 0.$$

Thus, $G_i^* - g_i^*$ depends only on r_i but not on v_i.

For the remaining parts of this Lemma, we only establish the statements for the upper best-response function; the proof for the corresponding part (b) is omitted.

Part 2. The continuity of Φ_i follow from the first-order condition and the fact that $f(r_i, 0) = 0$; it is differentiable as f is differentiable.

Part 3. Denote $\Phi_i(g) - g$ by y. For $g < v_i$, differentiate the first-order condition that defines the upper threshold with respect to g to get

$$1 + \frac{\partial f}{\partial y}\left(\frac{\mathrm{d}\Phi_i}{\mathrm{d}g} - 1\right) = 0,$$
$$\frac{\partial^2 f}{\partial y^2}\left(\frac{\mathrm{d}\Phi_i}{\mathrm{d}g} - 1\right)^2 + \frac{\partial f}{\partial y}\frac{\mathrm{d}^2\Phi_i}{\mathrm{d}g^2} = 0.$$

For $y > 0$, since

$$\frac{\partial f}{\partial y} = -R_1 R_2 \frac{e^{R_2 y} - e^{R_1 y}}{R_2 e^{R_1 y} - R_1 e^{R_2 y}} > 0,$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{R_1 R_2 (R_2 - R_1)^2 e^y}{(R_2 e^{R_1 y} - R_1 e^{R_2 y})^2} > 0,$$

we have $\mathrm{d}\Phi_i/\mathrm{d}g < 1$ and $\mathrm{d}^2\Phi_i/\mathrm{d}g^2 < 0$.

Part 4. The one-sided stopping problem satisfies a smooth pasting condition at the optimal threshold (see, for example, Dixit 1993). For any $g < v_i$ we have:

$$\left.\frac{\partial u_i}{\partial \theta}\right|_{(g,\Phi_i(g);\theta=\Phi_i(g))} = \left.\frac{\mathrm{d}(e^\theta/(1+e^\theta))}{\mathrm{d}\theta}\right|_{\theta=\Phi_i(g)}.$$

Differentiate both sides with respect to g to yield:

$$\left.\left(\frac{\mathrm{d}^2(e^\theta/(1+e^\theta))}{\mathrm{d}\theta^2} - \frac{\partial^2 u_i}{\partial\theta\partial G} - \frac{\partial^2 u_i}{\partial \theta^2}\right)\right|_{(g,\Phi_i(g);\theta=\Phi_i(g))} \frac{\mathrm{d}\Phi_i(g)}{\mathrm{d}g} = \left.\frac{\partial^2 u_i}{\partial\theta\partial g}\right|_{(g,\Phi_i(g);\theta=\Phi_i(g))}.$$

Recall from part 1 of Lemma 1 that the sign of $\partial u_i/\partial g$ depends on the sign of $f(r_i, G - g) - G + v_i$ and not on θ. In particular, $\partial u_i/\partial g = w_i(g, G; \theta)(e^{f(r_i,G-g)} - e^{G-v_i})$ for some positive function w_i. At $g = g_i^*$, we have $\Phi_i(g_i^*) = G_i^*$ and $f(r_i, G_i^* - g_i^*) - G_i^* + v_i = 0$, Thus,

$$\left.\frac{\partial^2 u_i}{\partial\theta\partial g}\right|_{(g_i^*,\Phi_i(g_i^*);\theta=\Phi_i(g_i^*))} = \left(e^{f(r_i,G_i^*-g_i^*)} - e^{G_i^*-v_i}\right)\left.\frac{\partial w_i}{\partial\theta}\right|_{(g_i^*,G_i^*;\theta=G_i^*)} = 0.$$

Thus, $\mathrm{d}\Phi_i(g_i^*)/\mathrm{d}g = 0$. Since Φ_i is strictly concave by part 3 of this lemma, Φ_i strictly increases in g when $g < g_i^*$ and strictly decreases in g when $g > g_i^*$.

Part 5. For $g \leq v_i$, $\Phi_i(g)$ satisfies $-g + f(r_i, \Phi_i(g) - g) - v_i = 0$. Since $f(r_i, y)$ is increasing in y, $\Phi_i(g)$ is increasing in v_i. For $g > v_i$, a marginal increase in v_i has no effect on $\Phi_i(g)$, while raising v_i to $v_i' > g$ will bring $\Phi_i(g)$ from g to a number strictly greater than g. For comparative statics with respect to r_i, we note from (5) that $\partial R_2/\partial r_i > 0$ and

$\partial R_1/\partial r_i = -\partial R_2/\partial r_i$. Therefore $\partial f/\partial r_i$ has the same sign as:

$$\begin{aligned}
&(e^{R_1 y} - R_2 y e^{R_1 y} + e^{R_2 y} - R_1 y e^{R_2 y})(R_2 - R_1) - 2(R_2 e^{R_1 y} - R_1 e^{R_2 y}) \\
=&(1 - (R_2 - R_1)R_1 y)e^{R_2 y} - (1 + (R_2 - R_1)R_2 y)e^{R_1 y} \\
=&(1 - (R_2 - R_1)R_1 y)e^{R_1 y}\left(e^{(R_2 - R_1)y} - \frac{1 + (R_2 - R_1)R_2 y}{1 - (R_2 - R_1)R_1 y}\right) \\
>&(1 - (R_2 - R_1)R_1 y)e^{R_1 y}\left(1 + (R_2 - R_1)y - \frac{1 + (R_2 - R_1)R_2 y}{1 - (R_2 - R_1)R_1 y}\right) \\
=& - R_1(R_2 - R_1)^2 y^2 e^{R_1 y} > 0.
\end{aligned}$$

Since $f(r_i, y)$ increases in both r_i and y, $\Phi_i(g)$ is decreasing in r_i for $g \leq v_i$. For $g > v_i$, $\Phi_i(g) = g$ is weakly decreasing in r_i.

Part 6. We know that $f(r_i, y)$ increases in y and $f(r_i, 0) = 0$. By L'Hopital's rule,

$$\lim_{r_i \to \infty} f(r_i, y) = \lim_{r_i \to \infty} \left(1 + \frac{r_i}{\mu'}\right)\left(e^{R_1 y} + e^{R_2 y}\right) = \infty.$$

We can choose $\bar{r}_i$ such that

$$v_i - g_0 - f(\bar{r}_i, \epsilon) = 0.$$

Since $f(r_i, y)$ increases in r_i, for any $r_i \geq \bar{r}_i$ and $g \in (g_0, v_i)$, we must have $\Phi_i(g) - g \leq \epsilon$. ∎

Proof of Lemma 3. For any $y > 0$, let $\gamma_i(y)$ be the value of g that solves $\Phi_i(g) - g = y$, where $i \in \{1, \ldots, 2m-1\}$ or $i = piv$. Similarly, let $\Gamma_i(y)$ be the value of G that solves $G - \phi_i(G) = y$. From the proof of Lemma 2, for $\Phi_i(g) - g > 0$ it must satisfy $v_i - g - f(r_i, \Phi_i(g) - g) = 0$ for $i \in \{1, \ldots, 2m-1\}$. Thus, for $y > 0$, the value of g that solves $\Phi_i(g) - g = y$ is given by

$$\gamma_i(y) = v_i - f(r_i, y).$$

Similarly, for $y > 0$, the value of G that solves $G - \phi_i(G) = y$ is

$$\Gamma_i(y) = v_i + f(r_i, y).$$

Part 3 of Lemma 2 implies that γ_i is strictly decreasing for each agent i. Hence γ_{piv} is also strictly decreasing. Similarly, Γ_i is strictly increasing for $i \in \{1, \ldots, 2m-1\}$ or $i = piv$.

Suppose there exists $\hat{y} > 0$ such that

$$\Gamma_{piv}(\hat{y}) - \gamma_{piv}(\hat{y}) = \hat{y}.$$

Then, from the definition of γ_{piv}, this equation implies $\Gamma_{piv}(\hat{y}) = \Phi_{piv}(\gamma_{piv}(\hat{y}))$. Further, using the definition of Γ_{piv}, we can also obtain $\gamma_{piv}(\hat{y}) = \phi_{piv}(\Gamma_{piv}(\hat{y}))$. In other words,

$(\gamma_{piv}(\hat{y}), \Gamma_{piv}(\hat{y}))$ is a fixed point of (ϕ_{piv}, Φ_{piv}). Moreover, since $\hat{y} > 0$, $\Gamma_{piv}(\hat{y}) > \gamma_{piv}(\hat{y})$. Therefore $(\gamma_{piv}(\hat{y}), \Gamma_{piv}(\hat{y}))$ belongs to $\hat{\Sigma}(k)$. We show the existence of $\hat{y}$ through two steps and the use of the intermediate value theorem.

Step 1. When y is sufficiently close to 0, $\Gamma_{piv}(y) - \gamma_{piv}(y) < y$.

Since $\lim_{y \to 0} \Gamma_{piv}(y) = v_{2m-k}$ and $\lim_{y \to 0} \gamma_{piv}(y) = v_k$, it follows from continuity of Γ_{piv} and γ_{piv} that the claim is true for y sufficiently close to 0 if $v_k > v_{2m-k}$. Now, suppose $v_k = v_{2m-k}$. By continuity, there exists some agent j with $v_j = v_k$ such that $\gamma_{piv}(y) = \gamma_j(y)$ for y sufficiently close to 0. Similarly, there is some agent j' with $v_{j'} = v_{2m-k}$ such that $\Gamma_{piv}(y) = \Gamma_{j'}(y)$ for y sufficiently close to 0. Therefore, for such y,

$$\Gamma_{piv}(y) - \gamma_{piv}(y) = v_{j'} - v_j + f(r_{j'}, y) + f(r_j, y).$$

Since $v_j = v_{j'}$ and $\partial f / \partial r > 0$, we have

$$\Gamma_{piv}(y) - \gamma_{piv}(y) \leq 2f\left(\max\{r_j, r_{j'}\}, y\right).$$

Let y_1 be the positive solution to the equation $-y + 2f(r, y) = 0$ when $r = \max\{r_j, r_{j'}\}$. We note that $-y + 2f(r, y)$ is strictly convex and is equal to 0 at $y = 0$ and $y = y_1$. Therefore $2f(r, y) < y$ for all $y \in (0, y_1)$, which implies $\Gamma_{piv}(y) - \gamma_{piv}(y) < y$ for y sufficiently close to 0.

Step 2. When y is sufficiently large, $\Gamma_{piv}(y) - \gamma_{piv}(y) > y$.

Let $r_{\min} = \min\{r_i \mid i = 1, \ldots, 2m-1\}$. Since $\partial f / \partial r > 0$, for all $i \leq k$, $\gamma_i(y) \leq v_k - f(r_{\min}, y)$ This implies that

$$\gamma_{piv}(y) \leq v_k - f(r_{\min}, y).$$

By the same token,

$$\Gamma_{piv}(y) \geq v_{2m-k} + f(r_{\min}, y).$$

Let y_2 be the positive solution to $-y + 2f(r, y) = 0$ when $r = r_{\min}$. Since $-y + 2f(r, y)$ is strictly convex and is equal to 0 at $y = 0$ and $y = y_2$, we have $2f(r, y) > y$ for all $y > y_2$. Moreover $v_k \geq v_{2m-k}$. Therefore, $\Gamma_{piv}(y) - \gamma_{piv}(y) > y$ for y sufficiently large. ∎

Proof of Lemma 4. From the definition of Ψ_i and ψ_i in (1) and (2), we can write the payoff $u_i(g, G \mid \theta)$ as:

$$\frac{e^{\theta}}{1+e^{\theta}}\left[\Psi_i(g', G' \mid \theta; A)\Psi_i(g, G \mid G'; A) + \psi_i(g', G' \mid \theta; A)\Psi_i(g, G \mid g'; A)\right]$$
$$+\frac{e^{v_i}}{1+e^{\theta}}\left[\psi_i(g', G' \mid \theta; B)\psi_i(g, G \mid g'; B) + \Psi_i(g', G' \mid \theta; B)\psi_i(g, G \mid G'; B)\right].$$

From Stokey (2009, Proposition 5.3),

$$\psi_i(g', G' \mid \theta; A) = \frac{e^{R_2(G'-\theta)} - e^{R_1(G'-\theta)}}{e^{R_2(G'-g')} - e^{R_1(G'-g')}} = e^{-(\theta-g')}\psi_i(g', G' \mid \theta; B),$$
$$\Psi_i(g', G' \mid \theta; B) = \frac{e^{R_2(\theta-g)} - e^{R_1(\theta-g)}}{e^{R_2(G-g)} - e^{R_1(G-g)}} = e^{-(G'-\theta)}\Psi_i(g', G' \mid \theta; A);$$

and the remaining terms are given by (3) and (4). Hence,

$$\begin{aligned}
&u_i(g, G \mid \theta) - u_i(g', G' \mid \theta) \\
&= \frac{e^{\theta}}{1+e^{\theta}}\left[(\Psi_i(g', G' \mid \theta; A)(\Psi_i(g, G \mid G'; A) - 1) + \psi_i(g', G' \mid \theta; A)\Psi_i(g, G \mid g'; A)\right] \\
&\quad + \frac{e^{v_i}}{1+e^{\theta}}\left[\psi_i(g', G' \mid \theta; B)(\psi_i(g, G \mid g'; B) - 1) + \Psi_i(g', G' \mid \theta; B)\psi_i(g, G \mid G'; B)\right] \\
&= \frac{1+e^{G'}}{1+e^{\theta}}e^{-(G'-\theta)}\Psi_i(g', G' \mid \theta; A)\left(u_i(g, G \mid G') - \frac{e^{G'}}{1+e^{G'}}\right) \\
&\quad + \frac{1+e^{g'}}{1+e^{\theta}}\psi_i(g', G' \mid \theta; B)\left(u_i(g, G \mid g') - \frac{e^{v_i}}{1+e^{g'}}\right).
\end{aligned}$$

For $\theta \in [g, G]$, $u_i(g, G \mid \theta)$ is increasing in G if $G \leq \Phi_i(g)$ and is decreasing in g if $g \geq \phi_i(G)$. When these latter two conditions are satisfied,

$$u_i(g, G \mid G') \geq u_i(g, G' \mid G') = e^{G'}/(1+e^{G'}),$$
$$u_i(g, G \mid g') \geq u_i(g', G \mid g') = e^{v_i}/(1+e^{g'}).$$

This proves $u_i(g, G \mid \theta) \geq u_i(g', G' \mid \theta)$. Similarly, $u_i(g, G \mid \theta)$ is decreasing in G if $G \geq G' \geq \Phi_i(g)$ and increasing in g if $g \leq g' \leq \phi_i(G)$. When these conditions are satisfied,

$$u_i(g, G \mid G') \leq u_i(g, G' \mid G') = e^{G'}/(1+e^{G'}),$$
$$u_i(g, G \mid g') \leq u_i(g', G \mid g') = e^{v_i}/(1+e^{g'}).$$

This proves $u_i(g, G \mid \theta) \leq u_i(g', G' \mid \theta)$. ■

Proof of Proposition 1. In the text, we show that any strategy profile σ that satisfies conditions 1–3 of Definition 1 must have the property that $(g^{[k]}(\sigma), G^{[k]}(\sigma)) \equiv (\hat{g}, \hat{G})$ belongs to the set $\hat{\Sigma}(k)$. We need to show that any strategy profile σ that satisfies conditions 1–3 of the proposition must be an equilibrium; i.e., no agent i can gain by deviating to another (g_i, G_i) within his span of control. The proof of the proposition proceeds in three steps: (1) show that $(\hat{g}, \hat{G})$ satisfies the Kuhn-Tucker necessary conditions for the solution of the constrained maximization problem (8) for each agent i and for every θ; (2) show that $(\hat{g}, \hat{G})$ is the only candidate solution to (8) that satisfies the Kuhn-Tucker necessary conditions; and (3) verify that $u_i(\hat{g}, \hat{G} \mid \theta)$ is greater than the payoff from immediate adoption of α or β when it is feasible. Finally, Lemma 3 establishes that $\hat{\Sigma}(k)$ is non-empty; hence equilibrium existence follows.

Step 1. Let $\underline{g}_i$ and $\overline{g}_i$ denote the lower and upper bounds of $I_g(i;\sigma)$, and $\underline{G}_i$ and $\overline{G}_i$ the lower and upper bounds of $I_G(i;\sigma)$. A necessary condition for $(\hat{g}, \hat{G})$ to solve problem (8) is that it must satisfy the following Kuhn-Tucker conditions:

$$\begin{cases} \partial u_i/\partial G|_{(\hat{g},\hat{G};\theta)} \leq 0 & \text{if } \hat{G} = \underline{G}_i, \\ \partial u_i/\partial G|_{(\hat{g},\hat{G};\theta)} = 0 & \text{if } \hat{G} \in (\underline{G}_i, \overline{G}_i), \\ \partial u_i/\partial G|_{(\hat{g},\hat{G};\theta)} \geq 0 & \text{if } \hat{G} = \overline{G}_i; \end{cases} \qquad \begin{cases} \partial u_i/\partial g|_{(\hat{g},\hat{G};\theta)} \leq 0 & \text{if } \hat{g} = \underline{g}_i, \\ \partial u_i/\partial g|_{(\hat{g},\hat{G};\theta)} = 0 & \text{if } \hat{g} \in (\underline{g}_i, \overline{g}_i), \\ \partial u_i/\partial g|_{(\hat{g},\hat{G};\theta)} \geq 0 & \text{if } \hat{g} = \overline{g}_i. \end{cases}$$

Note that any feasible boundary solutions (g, G) with either $g > \theta$ or $G < \theta$ would automatically satisfy the necessary conditions. In the first case β is adopted immediately; in the second, α is adopted immediately. Thus, the necessary conditions have bite only when $g \leq \theta \leq G$. By the dynamic consistency property of Lemma 1, $(\hat{g}, \hat{G})$ satisfies the necessary conditions for for all $\theta \in [\hat{g}, \hat{G}]$ if it satisfies the conditions for some $\theta \in (\hat{g}, \hat{G})$. Further, by the single-crossing property of Lemma 1 and from the definition of the the best-response functions, the necessary Kuhn-Tucker conditions can be re-stated as:

$$\begin{cases} \Phi_i(\hat{g}) \leq \hat{G} & \text{if } \hat{G} = \underline{G}_i, \\ \Phi_i(\hat{g}) = \hat{G} & \text{if } \hat{G} \in (\underline{G}_i, \overline{G}_i), \\ \Phi_i(\hat{g}) \geq \hat{G} & \text{if } \hat{G} = \overline{G}_i; \end{cases} \qquad \begin{cases} \phi_i(\hat{G}) \leq \hat{g} & \text{if } \hat{g} = \underline{g}_i, \\ \phi_i(\hat{G}) = \hat{g} & \text{if } \hat{g} \in (\underline{g}_i, \overline{g}_i), \\ \phi_i(\hat{G}) \geq \hat{g} & \text{if } \hat{g} = \overline{g}_i. \end{cases}$$

If $\Phi_i(\hat{g}) < \hat{G}$, condition 2(a) of Proposition 1 implies that $G_i < \hat{G}$. Thus the feasible span of control for agent i on the upper threshold is $I_G(i;\sigma) = [G^{[k]}(\sigma), G^{[k+1]}(\sigma)]$. In other words, we have $\hat{G} = \underline{G}_i$. This proves that last inequality on the left column of the Kuhn-Tucker conditions. If $\Phi_i(\hat{g}) > \hat{G}$, then condition 2(a) implies $G_i > \hat{G}$. Thus $I_G(i;\sigma) = [G^{[k-1]}(\sigma), G^{[k]}(\sigma)]$, and we have $\hat{G} = \overline{G}_i$. This proves the first inequality on the left column. Finally, if $\hat{G} \in (\underline{G}_i, \overline{G}_i)$, then $\Phi_i(\hat{g})$ is neither strictly greater nor strictly less than $\hat{G}$. Hence, the second inequality on the left column holds. By a similar argument for the lower threshold, $(\hat{g}, \hat{G})$ indeed satisfies the necessary Kuhn-Tucker conditions for the constrained maximization problem (8) for any belief $\theta \in [\hat{g}, \hat{G}]$.

Step 2. We now show that there is no other feasible threshold $(g', G') \in I_g(i;\sigma) \times I_G(i;\sigma)$ that satisfies the necessary Kuhn-Tucker conditions for belief $\theta \in [g', G']$. Suppose, by way of contradiction, such a $(g', G') \neq (\hat{g}, \hat{G})$ exists. Since the preference for the one-sided stopping problem is single-peaked, it must be that $g' \neq \hat{g}$ and $G' \neq \hat{G}$. Assume without loss of generality that $\hat{g} < g'$. Since both $\hat{g}$ and g' belong to $I_g(i;\sigma)$, we must have $\hat{g} < \overline{g}_i$ and $g' > \underline{g}_i$. The necessary Kuhn-Tucker conditions then imply that

$$\phi_i(G') \geq g' > \hat{g} \geq \phi_i(\hat{G}).$$

We claim that $\hat{G} \geq v_i$. Suppose the claim were not true and that $\hat{G} < v_i$. By Lemma 2, $\phi_i(\hat{G}) = \hat{G}$. As $\hat{g} \geq \phi_i(\hat{G})$ from the displayed inequalities above and $\hat{g} \leq \hat{G}$, we must have $\hat{g} = \hat{G} < v_i$. It follows that $\Phi_i(\hat{g}) > \hat{g} = \hat{G}$, and, by the Kuhn-Tucker conditions, $\hat{G} = \overline{G}_i$. In this case, $G' < \hat{G} < v_i$ and therefore $\phi_i(G') = G' < \hat{G} = \phi_i(\hat{G})$, which contradicts the displayed inequalities.

Since $\hat{G} \geq v_i$ is true, $\phi_i(\hat{G}) \geq g_i^*$. From the displayed inequalities, we obtain $g' \geq g_i^*$ and $\hat{g} \geq g_i^*$. Since the best-response functions are symmetric, the same argument implies that $G' \leq G_i^*$ and $\hat{G} \leq G_i^*$, and at least one of g' and $\hat{g}$ is less than or equal to v_i.

There remains two cases that we need to consider. First, suppose $\hat{G} < G'$. In this case, $v_i < G' \in (v_i, G_i^*]$. As ϕ_i is strictly decreasing between $(v_i, G_i^*]$, we would have $\phi_i(G') < \phi_i(\hat{G})$, which contradicts the displayed inequalities.

Finally, suppose $\hat{G} > G'$. The necessary Kuhn-Tucker conditions then imply $\Phi_i(\hat{g}) \geq \hat{G}$ and $\Phi_i(g') \leq G'$. Thus we have

$$G_i^* \geq \Phi_i(\hat{g}) \geq \hat{G} > G' \geq \Phi_i(g').$$

If $g' > v_i$, then $G' > v_i$. Since $G' \in (v_i, G_i^*)$ and ϕ_i is decreasing in this range, we have $\phi_i(G') < \phi_i(v_i) = v_i < g'$, which contradicts the inequality $\phi_i(G') \geq g'$. Hence, it must be the case that $g' \leq v_i$. By the same reasoning, we must have $G' \geq v_i$.

So far, we have shown that $g' \in (g_i^*, v_i]$ and $G' \in [v_i, G_i^*)$. From the two sets of inequalities displayed above, we also have

$$G' - g' \geq \Phi_i(g') - \phi_i(G').$$

Both the curve $(g, \Phi_i(g))$ and the line $g + G = 2v_i$ pass through (v_i, v_i) and (g_i^*, G_i^*). Since Φ_i is strictly concave (part 3 of Lemma 2), for any $g' \in (g_i^*, v_i]$, $\Phi_i(g') \geq 2v_i - g'$, with strict inequality if $g' \neq v_i$. Similarly, the strictly convexity of ϕ_i implies that for any $G' \in [v_i, G_i^*)$, $\phi_i(G') \leq 2v_i - G'$, with strict inequality if $G' \neq v_i$. Since $(g', G') \neq (v_i, v_i)$, we must have

$$\Phi_i(g') - \phi_i(G') > G' - g',$$

a contradiction.

Step 3. We have established that, for any θ, the only (g, G) that satisfies the necessary Kuhn-Tucker conditions with $\theta \in [g, G]$ is $(g, G) = (\hat{g}, \hat{G})$. Since a solution to the constrained maximization problem (8) always exists, if we can show that $u_i(\hat{g}, \hat{G} \mid \theta)$ is greater than $e^{v_i}/(1+e^\theta)$ when it is feasible to adopt β immediately at θ, and is greater than $e^\theta/(1+e^\theta)$ when it is feasible to adopt α immediately, then $(\hat{g}, \hat{G})$ must be a solution to (8).

When $\theta \geq \hat{G}$, α is adopted immediately under $(\hat{g}, \hat{G})$. Adopting β immediately is feasible only when $\theta \leq \overline{g}_i$. But in this case, $\hat{g}$ would have to be strictly less than $\overline{g}_i$ (as $\hat{G} > \hat{g}$), and the necessary Kuhn-Tucker conditions would require that $\partial u_i/\partial g|_{(\hat{g},\hat{G};\theta)} \leq 0$. This implies that $\hat{G} \geq v_i$, meaning that agent i would prefer α to β at θ.

When $\theta \leq \hat{g}$, β is chosen immediately under $(\hat{g}, \hat{G})$. Adopting α immediately is feasible only if $\theta \geq \underline{G}_i$. Following the above logic, this would require that $\partial u_i/\partial G|_{(\hat{g},\hat{G};\theta)} \geq 0$. This implies $\hat{g} \leq v_i$, and agent i would prefer β to α.

When $\theta \in (\hat{g}, \hat{G})$, adopting β immediately is feasible if $\theta \leq \overline{g}_i$. But in this case the necessary Kuhn-Tucker conditions imply that $\partial u_i/\partial g|_{(\hat{g},\hat{G};\theta)} \leq 0$ (as $\theta \geq \hat{g}$). It follows from Lemma 1 that $\partial u_i/\partial g|_{(g,\hat{G};\theta)} < 0$ for all $g > \hat{g}$. Thus $u_i(\hat{g}, \hat{G} \mid \theta) > u_i(\theta, \hat{G} \mid \theta) = e^{v_i}/(1+e^\theta)$. Adopting α immediately is feasible if $\theta > \underline{G}_i$. But in this case the necessary Kuhn-Tucker conditions imply that $\partial u_i/\partial G|_{(\hat{g},\hat{G};\theta)} \geq 0$, and it follows that $u_i(\hat{g}, \hat{G} \mid \theta) > u_i(\hat{g}, \theta \mid \theta) = e^\theta/(1+e^\theta)$. ■

Proof of Proposition 2. By part 5 of Lemma 2, when $v_1 = \ldots = v_{2m-1}$ and $r_1 \leq \ldots \leq r_{2m-1}$, $\Phi_{piv} = \Phi_k$ and $\phi_{piv} = \phi_k$. Hence $\hat{\Sigma}(k) = \{(g_k^*, G_k^*)\}$ is a singleton by part 1 of Lemma 2. When $r_1 = \ldots = r_{2m-1}$, $\Phi_{piv} = \Phi_k$ and $\phi_{piv} = \phi_{2m-k}$. Therefore, the equilibrium thresholds $(\hat{g}, \hat{G})$ must satisfy:

$$v_k - \hat{g} - f(r_k, \hat{G} - \hat{g}) = 0,$$
$$f(r_{2m-k}, \hat{G} - \hat{g}) - \hat{G} + v_{2m-k} = 0.$$

Let $\hat{y} = \hat{G} - \hat{g}$. These two equations reduce to

$$-\hat{y} + 2f(r_k, \hat{y}) = v_k - v_{2m-k}.$$

The function $-y + 2f(r_k, y)$ is convex and is equal to 0 when $y = 0$ and $y = G_k^* - g_k^*$. So $-y + 2f(r_k, y)$ is positive and strictly increasing for $y \geq G_k^* - g_k^*$. Since $v_k - v_{2m-k} \geq 0$, there is a unique $\hat{y} > 0$ that satisfies the equilibrium condition. The equilibrium lower threshold $\hat{g}$ satisfies $\Phi_k(\hat{g}) - \hat{g} = \hat{y}$, and since $\Phi_k(g) - g$ is strictly decreasing in g, there is a unique $\hat{g}$ associated with each $\hat{y}$. Similarly, since $\hat{G} - \phi_{2m-k}(\hat{G}) = \hat{y}$ and $G - \phi_{2m-k}(G)$ is strictly increasing in G, there is a unique $\hat{G}$ associated with each $\hat{y}$. Hence the set $\hat{\Sigma}(k)$ is a singleton.

In general, when agents differ in both preferences and discount rates, the model can admit multiple equilibria. Panel (a) of Figure 5 provides one example. Suppose both

$(\hat{g}, \hat{G})$ and $(\hat{g}', \hat{G}')$ are equilibria of the game. They must satisfy:

$$\Phi_{piv}(\hat{g}) - \hat{g} = \hat{G} - \phi_{piv}(\hat{G}),$$
$$\Phi_{piv}(\hat{g}') - \hat{g}' = \hat{G}' - \phi_{piv}(\hat{G}').$$

By Lemma 2, $\Phi_i(g) - g$ is positive and strictly decreasing if and only if $g < v_i$. Hence, $\Phi_{piv}(g) - g$ is positive and strictly decreasing if and only if $g < v_k$ under decision rule k. Similarly, $G - \phi_{piv}(G)$ is positive and strictly increasing if and only if $G > v_{2m-k}$. Proposition 1 already establishes that $G - g = \Phi_{piv}(g) - g > 0$ in any equilibrium. Hence, both $\hat{g}$ and $\hat{g}'$ are strictly lower than v_k. By a similar reasoning, $\hat{G}$ and $\hat{G}'$ are both strictly greater than v_{2m-k}. Therefore, by the two equations above, $\hat{g} > \hat{g}'$ implies $\hat{G} < \hat{G}'$, and $\hat{g} < \hat{g}'$ implies $\hat{G} > \hat{G}'$. Furthermore, when $k = m$, $v_k = v_{2m-k} = v_m$, so any equilibrium waiting region $(\hat{g}, \hat{G})$ must contain v_m in its interior. ■

Proof of Proposition 3. Since $\hat{G} = \Phi_{piv}(\hat{g})$, we have

$$\hat{G} = \min\{\theta \mid \#\{i|\Phi_i(\hat{g}) \leq \theta\} \geq k\} \leq \min\{\theta \mid \#\{i|G_i^* \leq \theta\} \geq k\} = G_{(k)}^*.$$

Similarly,

$$\hat{g} = \max\{\theta \mid \#\{i|\phi_i(\hat{G}) \geq \theta\} \geq k\} \geq \min\{\theta \mid \#\{i|g_i^* \geq \theta\} \geq k\} = g_{(k)}^*.$$ ■

Proof of Proposition 4. From the proof of Lemma 3, $\Gamma_{piv}(y) - \gamma_{piv}(y)$ is less than y for all y sufficiently close to 0 and is greater than y for all $y > y_2$. Hence, any parameter that lowers $\Gamma_{piv} - \gamma_{piv}$ will raise the largest and smallest solutions to the equation $\Gamma_{piv}(y) - \gamma_{piv}(y) = y$ (Milgrom and Roberts 1994). Since $\Phi_{piv}(g) - g$ is strictly decreasing for $g < v_k$, its inverse function $\gamma_{piv}(y)$ is strictly decreasing for $y > 0$. For any g, an increase in k increases $\Phi_{piv}(g) - g$, so it raises its inverse $\gamma_{piv}(y)$. Similarly, an increase in k lowers $\Gamma_{piv}(y)$. Hence $\overline{y}(\mathbf{v}, \mathbf{r}, k)$ and $\underline{y}(\mathbf{v}, \mathbf{r}, k)$ increase in k.

By the comparative statics result of Lemma 2, $\Phi_{piv}(g) - g$ increases while $G - \phi_{piv}(G)$ decreases in $\mathbf{r}$. Therefore $\Gamma_{piv}(y) - \gamma_{piv}(y)$ increases in $\mathbf{r}$, which implies that $\overline{y}(\mathbf{v}, \mathbf{r}, k)$ and $\underline{y}(\mathbf{v}, \mathbf{r}, k)$ decrease in $\mathbf{r}$. ■

Proof of Proposition 5. Pick some $\epsilon \leq \min\{v_{m+1} - v_m, v_m - v_{m-1}\}$. By part 6 of Lemma 2, there exists $\bar{r}_i$ such that $\Phi_i(g) - g \leq \epsilon$ for any $g \geq v_m - \epsilon$ and $G - \phi_i(G) \leq \epsilon$ for $G \leq v_m + \epsilon$ if $r_i \geq \bar{r}_i$. Set $\bar{r} = \max\{\bar{r}_1, \ldots, \bar{r}_{2m-1}\}$. We prove the proposition for the case $r_i \geq \bar{r}$ for some $i \leq m$. The complementary case if $i \geq m$ can be established in a similar manner.

For $j \geq m+1$, we have $\phi_j(v_m + \epsilon) = v_m + \epsilon$; and for $j \leq m$, we have $\phi_j(v_m + \epsilon) < v_m + \epsilon$. Furthermore, because $r_i \geq \bar{r}$,

$$v_m + \varepsilon > \phi_i(v_m + \epsilon) \geq v_m.$$

This means that given $G = v_m + \epsilon$, only $m - 1$ agents would vote to adopt β when the posterior belief is $v_m + \epsilon$, but m agents, or more, would vote for β at some posterior belief greater than v_m. Hence, $\phi_{piv}(v_m + \epsilon) \in [v_m, v_m + \epsilon)$. Let $\tilde{y} \equiv v_m + \epsilon - \phi_{piv}(v_m + \epsilon) > 0$. By definition, $\Gamma_{piv}(\tilde{y}) = v_m + \epsilon$. Furthermore, since $\lim_{y \to 0} \gamma_{piv}(y) = v_m$ under simple majority rule, and $\gamma_{piv}(y)$ is strictly decreasing in y for $y > 0$, we have $\gamma_{piv}(\tilde{y}) < v_m$. It follows that

$$\Gamma_{piv}(\tilde{y}) - \gamma_{piv}(\tilde{y}) > \epsilon \geq \tilde{y},$$

where the last inequality obtains because $\epsilon - \tilde{y} = \phi_{piv}(\tilde{y}) - v_m \geq 0$. Finally, from the proof of Lemma 3, we know that $\Gamma_{piv}(y) - \gamma_{piv}(y) < y$ when y is sufficiently close to 0. Hence, by continuity, there exists $\hat{y} \in (0, \tilde{y})$ that satisfies the equilibrium condition:

$$\Gamma_{piv}(\hat{y}) - \gamma_{piv}(\hat{y}) = \hat{y}.$$

Furthermore, the equilibrium thresholds satisfy $\hat{G} = \Gamma_{piv}(\hat{y}) \in (v_m, v_m + \epsilon]$ and $\hat{g} = \phi_{piv}(\hat{G}) \in (v_m - \epsilon, v_m]$. ■

Proof of Proposition 6. We prove a more general version of the proposition for any decision rule $k \geq m$, and show that there no equilibrium such that $\hat{G} - \hat{g} > \zeta$ if

$$\#\{i \mid r_i < \underline{r}(\zeta)\} \leq 2m - k - 1.$$

The statement of the proposition follows by setting $k = m$.

Pick any $\zeta > 0$. Choose $\underline{r}$ such that $-\zeta + 2f(\underline{r}, \zeta) = v_{2m-1} - v_1$. Since $-y + 2f(\underline{r}, y) - (v_{2m-1} - v_1)$ is strictly convex and is strictly less than 0 when $y = 0$ and equal to zero when $y = \zeta$, we have $2f(\underline{r}, y) - (v_{2m-1} - v_1) > y$ for all $y > \zeta$. For any $y > 0$, $\Gamma_i(y) = v_i + f(r_i, y)$ increases in v_i and r_i. Since there are at least k agents with discount rates greater than $\underline{r}$,

$$\Gamma_{piv}(y) \geq v_1 + f(\underline{r}, y).$$

Similarly, since $\gamma_i(y) = v_i - f(r_i, y)$ increases in v_i and decreases in r_i, and since there are at least k agents with discount rates greater than $\underline{r}$,

$$\gamma_{piv}(y) \leq v_{2m-1} - f(\underline{r}, y).$$

Consequently, for all $y > \zeta$,

$$\Gamma_{piv}(y) - \gamma_{piv}(y) \geq 2f(\underline{r}, y) - (v_1 - v_{2m-1}) > y.$$

It follows that any equilibrium $\hat{y}$ must satisfy $\hat{y} \leq \zeta$. ■

Proof of Proposition 7. Pick any $\epsilon > 0$. Choose $\tilde{r}$ such that $-\epsilon + 2f(\tilde{r}, \epsilon) = 0$ (that is, for an agent with discount rate $\tilde{r}$, the width of his optimal waiting region in a single-person decision problem is ϵ). Since $-y + 2f(\tilde{r}, y)$ is strictly convex and is equal to 0 when $y = 0$ and $y = \epsilon$, we have $-y + 2f(\tilde{r}, y) < 0$ for all $y < \epsilon$. Suppose $\#\{i \mid r_i \geq \tilde{r}\} \leq 2(k-m)$. We first show that there exists some $j \in \{2m-k, \ldots, k\}$ such that (a) $r_j < \tilde{r}$; (b) $\#\{i \leq j-1 \mid r_i \geq \tilde{r}\} \leq j + k - 2m$; and (c) $\#\{i \geq j+1 \mid r_i \geq \tilde{r}\} \leq k - j$.

Let $\eta_1 \equiv \#\{i < 2m-k \mid r_i \geq \tilde{r}\}$ denote the number of agents with discount rates greater than $\tilde{r}$ that are on the left of agent $2m-k$. Let $\eta_2 \equiv \#\{i > k \mid r_i \geq \tilde{r}\}$ denote the number of agents with discount rates greater than $\tilde{r}$ that are on the right of agent k. Since there are $2(k-m)+1$ agents in the group $\{2m-k, \ldots, k\}$ and there are at most $2(k-m)$ agents with discount rates greater than $\tilde{r}$, there must be at least one $j \in \{2m-k, \ldots, k\}$ such that $r_j < \tilde{r}$. This establishes (a). Next, the the number of agents between $2m-k$ and k (inclusive) with discount rates less than $\tilde{r}$ is at least

$$(2m-1) - 2(k-m) - (2m-k-1-\eta_1) - (2m-k-1-\eta_2) = \eta_1 + \eta_2 + 1.$$

Hence, there is some agent $j \in \{2m-k, \ldots, k\}$ such that $r_j < \tilde{r}$, and

$$\#\{i \in \{2m-k, \ldots, j-1\} \mid r_i < \tilde{r}\} \geq \eta_1,$$
$$\#\{i \in \{j+1, \ldots, k\} \mid r_i < \tilde{r}\} \geq \eta_2.$$

Since the number of agents between $2m-k$ and $j-1$ (inclusive) is $j-2m+k$, there is at most $j-2m+k-\eta_1$ agents in this group with discount rates greater than $\tilde{r}$. Hence, $\#\{i \leq j-1 \mid r_i \geq \tilde{r}\} \leq j+k-2m$. Similarly, $\#\{i \geq j+1 \mid r_i \geq \tilde{r}\} \leq k-j$. This establishes (b) and (c).

Consider j that satisfies (a), (b) and (c). For any $y > 0$, $\Gamma_i(y) = v_i + f(r_i, y)$ is increasing in v_i and increasing in r_i, and since there are at most $j-k+2m$ agents from 1 to j with discount rates greater than $\tilde{r}$, we have

$$\#\{i \leq j \mid \Gamma_i(y) \leq v_j + f(\tilde{r}, y)\} \geq 2m-k.$$

Therefore there are at most $(2m-1)-(2m-k) = k-1$ agents with $\Gamma_i(y) \geq y + 2f(\tilde{r}, y)$. We conclude that

$$\Gamma_{piv}(y) \leq y + f(\tilde{r}, y).$$

Similar reasoning also establishes that

$$\gamma_{piv}(y) \geq y - f(\tilde{r}, y).$$

Hence,

$$\Gamma_{piv}(y) - \gamma_{piv}(y) \leq 2f(\tilde{r}, y) < y$$

for all $y < \epsilon$. This means that the equilibrium $\hat{y}$ that satisfies $\Gamma_{piv}(\hat{y}) - \gamma_{piv}(\hat{y}) = \hat{y}$ must be such that $\hat{y} \geq \epsilon$. ∎

Proof of Proposition 8. We first prove inequality (10) in the text. Then we show that all agents (except agent 1) prefer equilibrium P^s to equilibrium P^3 in some interval of beliefs containing $(v_2 + v_3)/2$, and that there is no belief such that all agents (except agent 1) prefer P^3 to P^s. Finally the proof the completed by showing that equilibrium P^3 dominates P^2 and P^1.

Step 1. Equilibrium P^3 satisfies:

$$v_2 - \hat{g}^3 - f(r, \hat{G}^3 - \hat{g}^3) = 0,$$
$$f(r, \hat{G}^3 - \hat{g}^3) - \hat{G}^3 + v_3 = 0.$$

Subtracting the first equation from the second and letting $\hat{y}^3 = \hat{G}^3 - \hat{g}^3$, we obtain:

$$-\hat{y}^3 + 2f(r, \hat{y}^3) = -(v_3 - v_2).$$

Under super-majority rule, the equilibrium P^s must satisfy:

$$-\hat{y}^s + 2f(r, \hat{y}^s) = v_3 - v_2.$$

In the proof of Lemma 2, we show that the width y^* of the unconstrained optimal waiting region satisfies $-y^* + 2f(r, y^*) = 0$. Since $-y + 2f(r, y) < 0$ for $y < y^*$ and is strictly increasing for $y > y^*$, we have

$$\hat{y}^s > y^* > \hat{y}^3.$$

Furthermore, subtracting $-y^* + 2f(r, y^*) = 0$ from from the equation that defines $\hat{y}^3$ yields:

$$y^* - \hat{y}^3 + (v_3 - v_2) = 2\left(f(r, y^*) - f(r, \hat{y}^3)\right).$$

Since $f(r, y)$ is convex with $\partial f(r, y^*)/\partial y = 1$, and y^* is greater than $\hat{y}^3$, the right-hand-side is smaller than $2(y^* - \hat{y}^3)$, which gives

$$y^* - \hat{y}^3 > v_3 - v_2.$$

Similarly,

$$\hat{y}^s - y^* + (v_3 - v_2) = 2\left(f(r, \hat{y}^s) - f(r, y^*)\right).$$

Since $f(r, \cdot)$ is convex and $\hat{y}^s > y^*$, the right-hand-side is greater than $2(\hat{y}^s - y^*)$, which gives

$$\hat{y}^s - y^* < v_3 - v_2.$$

This proves inequality (10).

Step 2. Adding the two equilibrium conditions equilibrium P^3, it is straightforward to see that the equilibrium waiting region is centered around $\overline{v} = (v_2 + v_3)/2$. Similarly, the equilibrium waiting region associated with P^s is also centered around $\overline{v}$. Instead of writing utility as a function of the thresholds g and G, we can write it as a function of the center of the waiting region $\overline{v}$ and its width y, where $y = \hat{y}^s, \hat{y}^3$.

Using equations (3) and (4) and after some re-arrangement, we obtain:

$$u_j(g, G \mid \theta_0) = \frac{1}{1 + e^{\theta_0}} \frac{\tilde{q}_j(\overline{v}, y; \theta_0, v_j)}{q_j(y)},$$

where

$$q_j(y) = e^{-R_1 y/2} + e^{-R_2 y/2},$$

$$\tilde{q}_j(\overline{v}, y; \theta_0, v_j) = \frac{Q_1 e^{-R_1 y/2} - Q_2 e^{-R_2 y/2}}{e^{-R_1 y/2} - e^{-R_2 y/2}};$$

with $Q_1 = e^{v_j} e^{R_1(\theta_0 - \overline{v})} + e^{\overline{v}} e^{R_2(\theta_0 - \overline{v})}$ and $Q_2 = e^{\overline{v}} e^{R_1(\theta_0 - \overline{v})} + e^{v_j} e^{R_2(\theta_0 - \overline{v})}$.

When $\theta_0 = \overline{v}$, the utility of agent j is

$$u_j(g, G \mid \overline{v}) = \frac{1}{1 + e^{\overline{v}}} \frac{e^{v_j} + e^{\overline{v}}}{q_j(y)}.$$

The value of $q_j(y)$ decreases in y for $y < y^*$, reaches a minimum at $y = y^*$ and increases thereafter. At $y = y^*$, we have

$$\frac{\mathrm{d}q_j(y^*)}{\mathrm{d}y} = -\frac{R_1}{2} e^{-R_1 y^*/2} - \frac{R_2}{2} e^{-R_2 y^*/2} = 0.$$

Since $\hat{y}^s > y^* > \hat{y}^3$, the difference $q_j(\hat{y}^s) - q_j(\hat{y}^3)$ can be written as

$$\begin{aligned}
&\int_0^{\hat{y}^s - y^*} \frac{\mathrm{d}q_j(y^*+x)}{\mathrm{d}y}\,\mathrm{d}x + \int_0^{y^*-\hat{y}^3} \frac{\mathrm{d}q_j(y^*-x)}{\mathrm{d}y}\,\mathrm{d}x \\
&< \int_0^{y^*-\hat{y}^3} \left(\frac{\mathrm{d}q_j(y^*+x)}{\mathrm{d}y} + \frac{\mathrm{d}q_j(y^*-x)}{\mathrm{d}y}\right)\mathrm{d}x \\
&= \int_0^{y^*-\hat{y}^3} \frac{1}{2}R_2 e^{-R_2 y^*/2}\left((e^{-R_1 x/2} + e^{R_1 x/2}) - (e^{-R_2 x/2} + e^{R_2 x/2})\right)\mathrm{d}x < 0.
\end{aligned}$$

The first inequality follows from inequality (10) and the fact that $\mathrm{d}q_j(y^*+x)/\mathrm{d}y > 0$ for $x > 0$; the equality follows taking the derivative of q_j and using the condition $\mathrm{d}q_j(y^*)/\mathrm{d}y = 0$; and the final inequality follows from the fact that $R_2 > -R_1$ and $e^{-Rx} + e^{Rx}$ is increasing in R for $x > 0$. This demonstrates that at $\theta_0 = \overline{v}$ the payoff to any agent $j \neq 1$ is higher in equilibrium P^s than in equilibrium P^3.

Next, we show that agent 2 is strictly better off under P^s than under P^3 for all $\theta_0 \in [\hat{g}^3, \overline{v}]$. Note that

$$\frac{\partial \tilde{q}_2}{\partial y} = \frac{(R_2 - R_1)(e^{\overline{v}} - e^{v_2})(e^{R_1(\theta_0 - \overline{v})} - e^{R_2(\theta_0 - \overline{v})})}{2e^{y/2}(e^{-R_1 y/2} - e^{-R_2 y/2})^2}.$$

Thus, for $\theta_0 \in [\hat{g}^3, \overline{v}]$, $\hat{q}_2$ is increasing in y since $v_2 < \overline{v}$. For such θ_0, we therefore have

$$u_2(\hat{g}^s, \hat{G}^s \mid \theta_0) = \frac{1}{1+e^{\theta_0}} \frac{\tilde{q}_2(\overline{v}, \hat{y}^s; \theta_0, v_2)}{q_2(\hat{y}^s)} > \frac{1}{1+e^{\theta_0}} \frac{\tilde{q}_2(\overline{v}, \hat{y}^3; \theta_0, v_2)}{q_2(\hat{y}^3)} = u_2(\hat{g}^3, \hat{G}^3 \mid \theta_0).$$

By a symmetric argument, we can also show that agent 3 strictly prefers equilibrium P^s to P^3 for all $\theta_0 \in [\overline{v}, \hat{G}^3]$. By continuity, there is an interval which contains $\overline{v}$ such that both agents 2 and 3 strictly prefer super-majority rule to simple majority rule if the initial belief belongs to that interval.

Finally, suppose $\theta_0 \in (\hat{g}^s, \hat{g}^3)$. Agent 2 gets utility $u_2(\hat{g}^s, \hat{G}^s \mid \theta_0)$ under equilibrium P^s, while β would be adopted immediately under equilibrium P^3. Since $u_2(g, \hat{G}^s \mid \theta_0)$ is decreasing in g if $g < \phi_2(\hat{G}^s)$, we have

$$u_2(\hat{g}^s, \hat{G}^s \mid \theta_0) > u_2(\theta_0, \hat{G}^s \mid \theta_0) = e^{v_2}/(1+e^{\theta_0}).$$

Thus agent 2 prefers equilibrium P^s to equilibrium P^3 for this range of θ_0.

For $\theta_0 \leq \hat{g}^s$, equilibrium P^s and P^3 yields the same outcome.

Moreover we have already shown that agent 2 prefers P^s to P^3 for $\theta_0 \in [\hat{g}^3, \overline{v}]$. Thus agent 2 prefers P^s to P^3 for all $\theta_0 \leq \overline{v}$. Similarly agent 3 prefers P^s to P^3 for all $\theta_0 \geq \overline{v}$. Therefore, there is no θ_0 such that both agent 2 and agent 3 prefer equilibrium P^3 to equilibrium P^s.

Step 3. Let $(\hat{g}^l, \hat{G}^l)$ be the waiting region in corresponding to equilibrium P^l $(l = 1, 2, 3)$. By Proposition 2, we have $(\hat{g}^3, \hat{G}^3) \supset (\hat{g}^2, \hat{G}^2) \supset (\hat{g}^1, \hat{G}^1)$.

Take $(g, G) = (\hat{g}^3, \hat{G}^3)$ and take $(g', G') = (\hat{g}^l, \hat{G}^l)$ for $l = 1, 2$. In equilibrium P^3, the relevant pivotal functions are Φ_2 and ϕ_3. For agent 2, we have $\Phi_2(\hat{g}^3) = \hat{G}^3$ and $\phi_2(\hat{G}^3) < \phi_3(\hat{G}^3) = \hat{g}^3$. For agent 3, we have $\Phi_3(\hat{g}^3) > \Phi_2(\hat{g}^3) = \hat{G}^3$ and $\phi_3(\hat{G}^3) = \hat{g}^3$. Therefore, by Lemma 4, $u_j(\hat{g}^3, \hat{G}^3 \mid \theta_0) \geq u_j(\hat{g}^l, \hat{G}^l \mid \theta_0)$ for $j = 2, 3$ and for $\theta_0 \in [\hat{g}^l, \hat{G}^l]$.

Suppose $\theta_0 \in (\hat{G}^l, \hat{G}^3]$ $(l = 1, 2)$. Equilibrium P^3 entails some waiting while equilibrium P^l entails immediate adoption of α. For agent 2 we have $\Phi_2(\hat{g}^3) = \hat{G}^3$, which implies that $u_2(\hat{g}^3, G \mid \theta_0)$ is increasing in G for $G \leq \hat{G}^3$. Thus

$$u_2(\hat{g}^3, \hat{G}^3 \mid \theta_0) \geq u_2(\hat{g}^3, \theta_0 \mid \theta_0) = e^{\theta_0}/(1 + e^{\theta_0}).$$

We also have $\Phi_3(\hat{g}^3) > \hat{G}^3$, which implies that $u_3(\hat{g}^3, G \mid \theta_0)$ is increasing in G for $G \leq \hat{G}^3$. Therefore the same conclusion holds for agent 3.

Suppose $\theta_0 \in [\hat{g}^3, \hat{g}^l)$ $(l = 1, 2)$. Equilibrium P^3 entails some waiting while equilibrium P^l entails immediate adoption of β. For agent 2 we have $\phi_2(\hat{G}^3) < \hat{g}^3$, which implies that $u_2(g, \hat{G}^3 \mid \theta_0)$ is decreasing for $g \geq \hat{g}^3$. Therefore

$$u_2(\hat{g}^3, \hat{G}^3 \mid \theta_0) \geq u_2(\theta_0, \hat{G}^3 \mid \theta_0) = e^{v_2}/(1 + e^{\theta_0}).$$

For agent 3, $\phi_3(\hat{G}^3) = \hat{g}^3$, which implies $u_3(g, \hat{G}^3 \mid \theta)$ is decreasing for $g \geq \hat{g}^3$. Therefore the same conclusion holds.

Finally, for $\theta_0 > \hat{G}^3$ or $\theta_0 < \hat{g}^3$, the outcomes in equilibrium P^3 are the same as those in equilibria P^1 and P^2, so the equilibrium payoffs are equal. ■

Proof of Proposition 9. For $\theta_0 \geq \hat{G}$ or $\theta_0 \leq \hat{g}$, majority rule and super-majority rule yield the same outcome. There are four other cases to consider.

1. $\theta_0 \in [G^*_m, \hat{G}]$. For belief in this region, super-majority rule will lead to some waiting while majority rule leads to immediate adoption of α. By Lemma 2, for $j \leq m$.

$$\Phi_j(\hat{g}) \leq \Phi_m(\hat{g}) < G^*_m.$$

Since $u_j(\hat{g}, G \mid \theta_0)$ is decreasing in G if $G > \Phi_j(\hat{g})$, for $\theta_0 \in [G^*_m, \hat{G}]$ we have

$$u_j(\hat{g}, \hat{G} \mid \theta_0) < u_j(\hat{g}, \theta_0 \mid \theta_0) = e^{\theta_0}/(1 + e^{\theta_0}).$$

Therefore agents from 1 to m strictly prefer majority rule to super-majority rule.

2. $\theta_0 \in [v_m, G^*_m]$. As in the proof of Proposition **??**, we can write the utility of agent j in

terms of the center of the waiting region v_m and the width y:

$$u_j(g, G \mid \theta_0) = \frac{1}{1+e^{\theta_0}} \frac{\tilde{q}_j(v_m, y; \theta_0)}{q_j(y)},$$

where the function $\tilde{q}_j$ and q_j are defined in the proof of the earlier proposition. At $\theta_0 = v_m$, the utility of agent j is

$$u_j(g, G \mid v_m) = \frac{1}{1+e^{v_m}} \frac{v_j + v_m}{q_j(y)}.$$

Since $q_j(y)$ is decreasing in y for $y > y^*$, we have $u_j(g^*, G^* \mid v_m) > u_j(\hat{g}, \hat{G} \mid v_m)$ for $j = 1, \ldots, 2m-1$. Since $\tilde{q}_j$ is decreasing in y for $\theta_0 > v_m$ and $v_j < v_m$, $\hat{y} > y^*$ implies $u_j(g^*, G^* \mid \theta_0) > u_j(\hat{g}, \hat{G} \mid \theta_0)$ for $j = 1, \ldots, m$.

3. $\theta_0 \in [g_m^*, v_m]$. Recall that $u_j(g^*, G^* \mid v_m) > u_j(\hat{g}, \hat{G} \mid v_m)$ for $j = 1, \ldots, 2m-1$. Since $\tilde{q}_j$ is decreasing in y for $\theta_0 < v_m$ and $v_j > v_m$, and since $q_j(y)$ is decreasing in y for $y > y^*$, $\hat{y} > y^*$ implies $u_j(g^*, G^* \mid \theta_0) > u_j(\hat{g}, \hat{G} \mid \theta_0)$ for $j = m, \ldots, 2m-1$.

4. $\theta_0 \in [\hat{g}, g_m^*]$. Super-majority rule will lead to some waiting while majority rule leads to immediate adoption of β. By Lemma 2, for $j \geq m$, $\phi_j(\hat{G}) \geq \phi_m(\hat{G}) > g_m^*$. Since $u_j(g, \hat{G} \mid \theta_0)$ is increasing in g if $g < \phi_j(\hat{G})$, for $\theta_0 \in [\hat{g}, g_m^*]$ we have $u_j(\hat{g}, \hat{G} \mid \theta_0) < e^{v_j}/(1+e^{\theta_0})$. Therefore agents from m to $2m-1$ strictly support majority rule.

Note from cases 2 and 3 above that all agents strictly prefer majority rule to super-majority rule when $\theta_0 = v_m$. Since preferences are continuous in θ_0, the second part of the proposition follows. ∎

Proof of Proposition 10. Let $k > m$, and let the equilibrium thresholds under super-majority rule be represented by g_k^* and G_k^*. Similarly, let the equilibrium thresholds under majority rule be represented by g_m^* and G_m^*. For $\theta_0 \geq G_k^*$ or $\theta_0 \leq g_k^*$, majority rule and super-majority rule yield the same outcome. There are three other cases to consider.

1. $\theta_0 \in [G_m^*, G_k^*]$. For belief in this region, super-majority rule will lead to some waiting while majority rule leads to immediate adoption of α. We have, for $j \leq m$,

$$\Phi_j(g_k^*) \leq \Phi_m(g_k^*) \leq \Phi_m(g_m^*) = G_m^*,$$

where the first inequality follows from the comparative statics property of Lemma 2 and $r_j \geq r_m$, while the second inequality follows from the non-monotonicity property of Lemma 2 and $g_k^* < g_m^*$. Since $u_j(g_k^*, G \mid \theta_0)$ is decreasing in G if $G \geq G_m^* \geq \Phi_j(g_k^*)$, for $\theta_0 \in [G_m^*, G_k^*]$ we have

$$u_j(g_k^*, G_k^* \mid \theta_0) \leq u_j(g_k^*, \theta_0 \mid \theta_0) = e^{\theta_0}/(1+e^{\theta_0}).$$

Therefore agents from 1 to m prefer majority rule to super-majority rule.

2. $\theta_0 \in [g_k^*, g_m^*]$. Super-majority rule will lead to some waiting while majority rule leads to immediate adoption of β. By the same reasoning as in case 1, for $j \leq m$ we have

$$\phi_j(G_k^*) \geq \phi_m(G_k^*) \geq \phi_m(G_m^*) = g_m^*,$$

which implies $u_j(g_k^*, G_k^* \mid \theta_0) \leq e^{v_j}/(1+e^{\theta_0})$. Therefore agents from 1 to m support majority rule.

3. $\theta_0 \in [g_m^*, G_m^*]$. In case 1 and case 2, we have already shown that for $j \leq m$, $\Phi_j(g_k^*) \leq G_m^*$ and $\phi_j(G_k^*) \geq g_m^*$. By Lemma 4, we obtain

$$u_j(g_k^*, G_k^* \mid \theta_0) \leq u_j(g_m^*, G_m^* \mid \theta_0).$$

Therefore agents from 1 to m prefer majority rule to super-majority rule. ■

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