Liquidity, Futures Price Dynamics, and Risk Management

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This paper examines the optimal design of a futures hedge program by a competitive firm under output price uncertainty. Due to a capital constraint and the marking-to-market procedure of futures contracts, the firm faces endogenous liquidity risk. If the futures prices are sufficiently positively correlated, we show that the capital constraint is non-binding in that the optimal amount of capital earmarked to the futures hedge program is less than the firm's capital endowment. Otherwise, we show that the capital constraint becomes binding in that the firm optimally puts aside all of its capital stock for the futures hedge program. In the case of non-binding capital constraint, we show that the firm's optimal futures position is likely to be an over-hedge for reasonable preferences. In the case of binding capital constraint, the firm's optimal futures position is an under-hedge or an over-hedge, depending on whether the autocorrelation coefficient of the futures price dynamics is below or above a critical positive value, respectively.

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Abstract

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1. Introduction

According to the Committee on Payment and Settlement Systems (1998), firms should take liquidity risk seriously when devising their risk management strategies. Failure to do so is likely to result in fatal consequences for even technically solvent firms. An apposite example of this sort is the disaster at Metallgesellschaft A. G. (MG), the 14th largest industrial firm in Germany.[1]

In 1993, MG Refining and Marketing, Inc. (MGRM), the U.S. subsidiary of MG, offered long-term contracts for oil and refined oil products that allow its customers to lock in fixed

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[1]Another example is the debacle of Long-Term Capital Management (Jorion, 2001).

prices up to 10 years into the future. To hedge its exposure to the oil price risk, MGRM took on large positions in energy derivatives, primarily in oil futures. When oil prices plummeted in December 1993, MGRM was unable to meet its variation margin payments due to the denial of credit from its banks.[2] This debacle resulted in a $2.4 billion rescue package coupled with a premature liquidation of its futures positions *en masse* so as to keep MG from going bankrupt (Culp and Miller, 1995).

While basis risk in oil futures would certainly imply that MG's hedge did not successfully lock in value (Ross, 1997; Hilliard, 1999; Neuberger, 1999), Mello and Parsons (1995, 2000) identify the funding requirements of MG's hedging strategy as one of the central causes of the problem. Indeed, Mello and Parsons (1995, 2000) show that a perfect hedge does not create its own liquidity, and that the inability to fund a hedging strategy to its end is a serious defect in the design of many popular hedging strategies. In light of these findings, the purpose of this paper is to examine whether there is any role of liquidity constraints in the optimal design of a futures hedge program that allows an endogenously determined provision for terminating the program.

This paper develops a two-period model of the competitive firm under output price uncertainty (Sandmo, 1971). Specifically, the firm produces a single commodity that is sold at the end of the planning horizon. Since the subsequent spot output price is not known ex ante, the firm trades unbiased futures contracts for hedging purposes. All of the unbiased futures contracts are marked to market in that they require cash settlements of gains and losses at the end of each period. The futures price dynamics is assumed to follow a first-order autoregression that includes a random walk as a special case.[3]

The firm devises its futures hedge program by choosing a futures position and an amount

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[2]Culp and Hanke (1994) report that "four major European banks called in their outstanding loans to MGRM when its problems became public in December 1993. Those loans, which the banks had previously rolled-over each month, denied MGRM much needed cash to finance its variation margin payments and exacerbated its liquidity problems."

[3]Using a unique data set of 280 different commodities, Andersson (2007) does not reject a unit root (random walk) except for some 15% of the commodity price series. He attributes these findings to the low power of statistical unit root tests. As an alternative to statistical tests, he proposes using the hedging error in option prices as an economic test of mean reversion. For the 162 series, the mean reverting process provides a mean absolute error of 3.8% compared to 7.5% for the geometric Brownian motion (random walk).

of capital earmarked for the program. According to its futures hedge program, the firm commits to premature liquidation of its futures position on which the interim loss incurred exhausts the earmarked capital. The capital commitment as such constitutes an endogenous liquidity constraint, where the choice of the former dictates the severity of the latter. The firm is subject to a capital constraint in that the earmarked capital cannot exceed the firm's capital endowment. We show that the liquidity risk arising from the capital constraint and the marking-to-market procedure of the futures contracts truncates the firm's payoff profile, which plays a pivotal role in shaping the optimal design of the firm's futures hedge program. Given that the futures prices are autocorrelated, the resulting intertemporal linkage is likely to induce the firm to consider a provision for premature termination of its futures hedge program.

In the benchmark case that the firm is not liquidity constrained, the celebrated separation and full-hedging theorems of Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980) apply. The separation theorem states that the firm's production decision depends neither on its risk attitude nor on the underlying output price uncertainty. The full-hedging theorem states that the firm should completely eliminate its output price risk exposure by adopting a full-hedge via the unbiased futures contracts.

When the choice of the capital commitment, i.e., the severity of the liquidity constraint, is endogenously determined by the firm, we show that the firm voluntarily chooses to limit the amount of capital earmarked for its futures hedge program should the futures prices be positively autocorrelated. The autoregressive specification of the futures price dynamics renders predictability, of which the firm has incentives to take advantage. Specifically, a positive autoregression implies that a loss from a futures position tends to be followed by another loss from the same position. The firm as such finds premature liquidation of its futures position to be ex-post optimal. The amount of capital earmarked to the futures hedge program is thus chosen to strike a balance between ex-ante and ex-post efficient risk sharing. If the futures prices are sufficiently positively correlated, we show that the capital constraint is non-binding in that the optimal capital commitment is less than the firm's capital endowment. Otherwise, we show that the capital constraint becomes

binding in that the firm optimally puts aside all of its capital stock for the futures hedge program. In the case of non-binding capital constraint, we show that the firm's optimal futures position is likely to be an over-hedge for reasonable preferences. In the case of binding capital constraint, the firm's optimal futures position is an under-hedge or an over-hedge, depending on whether the autocorrelation coefficient of the futures price dynamics is below or above a critical positive value, respectively. Finally, if the futures prices are uncorrelated or negatively autocorrelated, premature liquidation of the futures position is never ex-post optimal, thereby making the firm prefer not to be liquidity constrained and the separation and full-hedging theorems follow.

In a similar model in which the competitive firm faces an exogenous liquidity constraint and the futures price dynamics follows a random walk, Lien (2003) shows the optimality of an under-hedge. Wong (2004a, 2004b) and Wong and Xu (2006) further show that the liquidity constrained firm optimally cuts down its production. These results are in line with those of Paroush and Wolf (1989) in that the presence of residual unhedgeable risk would adversely affect the hedging and production decisions of the competitive firm under output price uncertainty. In contrast, we follow Wong (2008) to allow not only an endogenous liquidity constraint but also a first-order autocorrelation of the futures price dynamics. The latter renders the futures prices predictability of which the firm has incentives to take advantage. This explains why an over-hedge, coupled with a commitment to premature liquidation, is optimal when the futures prices are positively autocorrelated. When the futures prices are uncorrelated or negatively autocorrelated, premature liquidation is suboptimal and thus the firm adopts a full-hedge. In the case of an exogenously liquidity constraint, premature liquidation is inevitable so that an under-hedge is called for to limit the potential loss due to a lack of liquidity. The disparate results thus identify factors such as the predictability of futures prices, the severity of liquidity constraints, and the attitude of risk preferences to be crucial for the optimal design of a futures hedge program.

The rest of this paper is organized as follows. Section 2 develops a two-period model of the optimal design of a futures hedge program by a competitive firm under output price uncertainty. Due to a capital constraint and the marking-to-market procedure of futures

contracts, the firm faces endogenous liquidity risk. Section 3 examines a benchmark case in which the firm is not subject to any liquidity constraints. Section 4 derives the firm's optimal futures hedge program when the firm is endowed with an infinite amount of capital. Section 5 goes on to derive the firm's optimal futures hedge program when the firm is endowed with a finite amount of capital. Section 6 constructs numerical examples to shed light on the theoretical findings. The final section concludes.

2. The model

Consider a dynamic variant model of the competitive firm under output price uncertainty *à la* Sandmo (1971). There are two periods with three dates, indexed by $t = 0$, 1, and 2. Interest rates in both periods are known with certainty at $t = 0$. To simplify notation, we suppress the known interest factors by compounding all cash flows to their futures values at $t = 2$.

To begin, the firm is endowed with a fixed amount of capital, $\overline{k} > 0$. The firm produces a single commodity according to a deterministic cost function, $c(q)$, where $q \geq 0$ is the level of output chosen by the firm at $t = 0$. We assume that $c(q)$ satisfies that $c(0) = c'(0) = 0$, and that $c'(q) > 0$ and $c''(q) > 0$ for all $q > 0$. The firm sells its entire output, q, at $t = 2$ at the then prevailing spot price, p_2, that is not known ex ante.

To hedge its exposure to the output price uncertainty, the firm can trade infinitely divisible futures contracts at $t = 0$. Each of the futures contracts calls for delivery of one unit of output at $t = 2$, and is marked to market at $t = 1$. Let f_t be the futures price at date t ($t = 0$, 1, and 2). While the initial futures price, f_0, is predetermined at $t = 0$, the other futures prices, f_1 and f_2, are regarded as positive random variables. In the absence of basis risk, the futures price at $t = 2$ must be set equal to the spot price at that time by convergence. Thus, we have $f_2 = p_2$.

We model the futures price dynamics by assuming that $f_t = f_{t-1} + \varepsilon_t$ for $t = 1$ and 2,

where $\varepsilon_2 = \rho\varepsilon_1 + \delta$, ρ is a scalar, and ε_1 and δ are two random variables independent of each other. To focus on the firm's hedging motive, vis-à-vis its speculative motive, we further assume that ε_1 and δ have means of zero so that the initial futures price, f_0, is unbiased and set equal to the unconditional expected value of the random spot price at $t = 2$, p_2. The futures price dynamics as such is a first-order positive or negative autoregression, depending on whether ρ is positive or negative, respectively. If $\rho = 0$, the futures price dynamics becomes a random walk.

We delineate the firm's futures hedge program by a pair, (h, k), where $h > 0$ is the number of the futures contracts sold by the firm at $t = 0$, and $k \in [0, \overline{k}]$ is the fixed amount of capital earmarked for the futures hedge program.[4] Due to marking to market at $t = 1$, the firm suffers a loss (or enjoys a gain if negative) of $(f_1 - f_0)h$ at $t = 1$ from its short futures position, h. The firm's futures hedge program, (h, k), dictates the firm to prematurely liquidate its short futures position at $t = 1$ if the interim loss exhausts the earmarked capital, i.e., if $(f_1 - f_0)h > k$. Thus, the firm's random profit at $t = 2$ in this liquidation case, π_ℓ, is given by

$$\pi_\ell = p_2 q + (f_0 - f_1)h - c(q) = f_0 q + \varepsilon_1[(1 + \rho)q - h] + \delta q - c(q), \tag{1}$$

where the second equality follows from the assumed futures price dynamics. On the other hand, if $(f_1 - f_0)h \leq k$, the firm holds its short futures position until $t = 2$. Thus, the firm's random profit at $t = 2$ in this continuation case, π_c, is given by

$$\pi_c = p_2 q + (f_0 - f_2)h - c(q) = f_0 q + [(1 + \rho)\varepsilon_1 + \delta](q - h) - c(q), \tag{2}$$

where the second equality follows from the assumed futures price dynamics.

The firm is risk averse and possesses a von Neumann-Morgenstern utility function, $u(\pi)$, defined over its profit at $t = 2$, π, where $u'(\pi) > 0$ and $u''(\pi) < 0$.[5] Anticipating the

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[4]In the appendix, we show that it is never optimal for the firm to opt for a long futures position, i.e., $h < 0$. Hence, we can restrict our attention to the case that the firm always chooses a short futures position.

[5]The risk-averse behavior of the firm can be motivated by managerial risk aversion (Stulz, 1984), corporate taxes (Smith and Stulz, 1985), costs of financial distress (Smith and Stulz, 1985), and capital market imperfections (Stulz, 1990; Froot, Scharfstein, and Stein, 1993). See Tufano (1996) for evidence that managerial risk aversion is a rationale for corporate risk management in the gold mining industry.

endogenous liquidity constraint at $t = 1$, the firm chooses its output level, $q \geq 0$, and devises its futures hedge program, (h, k), so as to maximize the expected utility of its random profit at $t = 2$:

$$\max_{q \geq 0, h > 0, 0 \leq k \leq \overline{k}} \int_{-\infty}^{k/h} \mathrm{E}[u(\pi_c)]g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}[u(\pi_\ell)]g(\varepsilon_1) \, \mathrm{d}\varepsilon_1, \tag{3}$$

where $\mathrm{E}(\cdot)$ is the expectation operator with respect to the probability density function of δ, $g(\varepsilon_1)$ is the probability density function of ε_1, and π_ℓ and π_c are defined in Eqs. (1) and (2), respectively. We refer to the short futures position, h, as an under-hedge, a full-hedge, or an over-hedge if h is less than, equal to, or greater than q, respectively.

The Kuhn-Tucker conditions for program (3) are given by[6]

$$\int_{-\infty}^{k^*/h^*} \mathrm{E}\{u'(\pi_c^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1$$

$$+ \int_{k^*/h^*}^{\infty} \mathrm{E}\{u'(\pi_\ell^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 = 0, \tag{4}$$

$$- \int_{-\infty}^{k^*/h^*} \mathrm{E}\{u'(\pi_c^*)[(1+\rho)\varepsilon_1 + \delta]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 - \int_{k^*/h^*}^{\infty} \mathrm{E}[u'(\pi_\ell^*)]\varepsilon_1 g(\varepsilon_1) \, \mathrm{d}\varepsilon_1$$

$$- \mathrm{E}[u(\pi_{c0}^*) - u(\pi_{\ell0}^*)]g(k^*/h^*)k^*/h^{*2} = 0, \tag{5}$$

$$\mathrm{E}[u(\pi_{c0}^*) - u(\pi_{\ell0}^*)]g(k^*/h^*)/h^* - \lambda^* \leq 0, \tag{6}$$

and

$$\overline{k} - k^* \geq 0, \tag{7}$$

where conditions (5) and (6) follow from using Leibniz's rule, λ^* is the Lagrange multiplier, $\pi_{c0}^* = f_0 q^* + [(1+\rho)k^*/h^* + \delta](q^* - h^*) - c(q^*)$, $\pi_{\ell0}^* = [f_0 + (1+\rho)k^*/h^* + \delta]q^* - k^* - c(q^*)$, and an asterisk (*) signifies an optimal level. Should $k^* > 0$, condition (6) holds with equality. Likewise, should $\lambda^* > 0$, condition (7) holds with equality.

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[6]The second-order conditions for program (3) are satisfied given risk aversion and the strictly convexity of $c(q)$.

3.   Benchmark case with no liquidity constraints

As a benchmark, we consider in this section the case that the firm is not liquidity constrained, which is tantamount to setting $k = \infty$. In this benchmark case, Program (3) becomes

$$\max_{q \geq 0, h > 0} \int_{-\infty}^{\infty} \mathrm{E}[u(\pi_c)]g(\varepsilon_1) \, \mathrm{d}\varepsilon_1. \tag{8}$$

The first-order conditions for program (8) are given by

$$\int_{-\infty}^{\infty} \mathrm{E}\{u'(\pi_c^0)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^0)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 = 0, \tag{9}$$

and

$$-\int_{-\infty}^{\infty} \mathrm{E}\{u'(\pi_c^0)[(1+\rho)\varepsilon_1 + \delta]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 = 0, \tag{10}$$

where a nought $(^0)$ indicates an optimal level.

Solving Eqs. (9) and (10) yields the following proposition, where all proofs of propositions are given in Appendix B.

Proposition 1.   *Given that the competitive firm is not liquidity constrained, i.e., $k = \infty$, the firm's optimal output level, q^0, solves*

$$c'(q^0) = f_0, \tag{11}$$

and its optimal futures position, h^0, is a full-hedge, i.e., $h^0 = q^0$.

The intuition of Proposition 1 is as follows. If the firm is not liquidity constrained, its random profit at $t = 2$ is given by Eq. (2) only. The firm could have completely eliminated all the price risk had it chosen $h = q$ within its own discretion. Alternatively put, the degree of price risk exposure to be assumed by the firm should be totally unrelated to its production decision. The optimal output level is then chosen to maximize $f_0 q - c(q)$,

thereby yielding q^0 that solves Eq. (11). Since the futures contracts are unbiased, they offers actuarially fair "insurance" to the firm. Being risk averse, the firm finds it optimal to opt for full insurance via a full-hedge, i.e., $h^0 = q^0$. These results are simply the well-known separation and full-hedging theorems of Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980).

4. The case of infinite capital endowments

In this section, we consider the case that the firm is endowed with an infinite amount of capital, $\overline{k} = \infty$, and optimally chooses the liquidity threshold, k, at $t = 0$. This is also the case analyzed by Wong (2008).

The following proposition characterizes the firm's optimal liquidation threshold, k^*, as a function of the autocorrelation coefficient, ρ.

Proposition 2.   *Given that the competitive firm is endowed with an infinite amount of capital, $\overline{k} = \infty$, and optimally devises its futures hedge program, (h^*, k^*), the firm commits to the optimal liquidation threshold, k^*, that is positive and finite (infinite) if the autocorrelation coefficient, ρ, is positive (non-positive).*

The intuition of Proposition 2 is as follows. If the firm chooses $k = \infty$, risk sharing is ex-ante efficient because the firm can completely eliminate all the price risk. However, this is not ex-post efficient, especially when $\rho > 0$. To see this, note that for any given $k < \infty$ the firm prematurely liquidates its futures position at $t = 1$ for all $\varepsilon_1 \in [k/h, \infty)$. Conditioned on premature liquidation, the expected value of f_2 is equal to $f_1 + \rho\varepsilon_1$, which is greater (not greater) than f_1 when $\rho > (\leq) 0$. Thus, it is ex-post optimal for the firm to liquidate its futures position prematurely to limit further losses if $\rho > 0$. In this case, the firm chooses the optimal threshold level, k^*, to be finite so as to strike a balance between ex-ante and ex-post efficient risk sharing. If $\rho \leq 0$, premature liquidation is never ex-post

optimal and thus the firm chooses $k^* = \infty$.

The following proposition is an immediate consequence of Propositions 1 and 2.

Proposition 3. *Given that the competitive firm is endowed with an infinite amount of capital, $\bar{k} = \infty$, and optimally devises its futures hedge program, (h^*, k^*), the firm's optimal output level, q^*, equals the benchmark level, q^0, and its optimal futures position, h^*, is a full-hedge, i.e., $h^* = q^*$, if the autocorrelation coefficient, ρ, is non-positive.*

When $\rho > 0$, Proposition 2 implies that the firm voluntarily chooses to be liquidity constrained, i.e., $k^* < \infty$. Hence, in this case, condition (6) holds with equality and the solution, (q^*, h^*, k^*), solves Eqs. (4), (5), and (6) simultaneously. The following proposition characterizes the liquidity constrained firm's optimal futures position, h^*.

Proposition 4. *Given that the competitive firm is endowed with an infinite amount of capital, $\bar{k} = \infty$, and optimally devises its futures hedge program, (h^*, k^*), the firm's optimal futures position, h^*, is an over-hedge, i.e., $h^* > q^*$, if the autocorrelation coefficient, ρ, is positive and the firm's utility function, $u(\pi)$, satisfies either constant or increasing absolute risk aversion.*

To see the intuition of Proposition 4, we refer to Eqs. (1) and (2). If the firm adopts a full-hedge, i.e., $h = q$, its profit at $t = 2$ remains stochastic due to the residual price risk, $(\rho\varepsilon_1 + \delta)q$, that arises from the premature closure of its hedge program at $t = 1$. This creates an income effect because the presence of the liquidity risk reduces the attainable expected utility under risk aversion. To attain the former expected utility level (with no risk), the firm has to be compensated with additional income. Taking away this compensation gives rise to the income effect (see Wong, 1997). Under IARA (DARA), the firm becomes less (more) risk averse and thus is willing (unwilling) to take on the liquidity risk. The firm as such shorts more (less) of the futures contracts so as to enlarge (shrink) the interval, $[k/h, \infty)$, over which the premature liquidation of the futures position at $t = 1$ prevails.

Since $\rho > 0$, inspection of Eqs. (1) and (2) reveals that the high (low) realizations of the firm's random profit at $t = 2$ occur when the futures position is (is not) prematurely liquidated at $t = 1$. Being risk averse, the firm would like to shift profits from the high-profit states to the low-profit states. This goal can be achieved by shorting more of the futures contracts, i.e., $h > q$, as is evident from Eqs. (1) and (2). Such an over-hedging incentive is reinforced (alleviated) under IARA (DARA). Thus, the firm optimally opts for an over-hedge, i.e., $h^* > q^*$, under either CARA or IARA.

5. The case of finite capital endowments

In this section, we consider the case that the firm is endowed with a finite amount of capital, $0 < \bar{k} < \infty$, and optimally chooses the liquidity threshold, k, at $t = 0$.

Suppose that the capital constraint is strictly binding, i.e., $\lambda^* > 0$ so that $k^* = \bar{k}$.[7] The Kuhn-Tucker conditions for program (3) under the binding capital constraint become

$$\int_{-\infty}^{\bar{k}/h^*} \mathrm{E}\{u'(\pi_c^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$+ \int_{\bar{k}/h^*}^{\infty} \mathrm{E}\{u'(\pi_\ell^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 = 0, \tag{12}$$

$$- \int_{-\infty}^{\bar{k}/h^*} \mathrm{E}\{u'(\pi_c^*)[(1+\rho)\varepsilon_1 + \delta]\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 - \int_{\bar{k}/h^*}^{\infty} \mathrm{E}[u'(\pi_\ell^*)]\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$- \mathrm{E}[u(\pi_{c0}^*) - u(\pi_{\ell0}^*)]g(\bar{k}/h^*)\bar{k}/h^{*2} = 0, \tag{13}$$

and

$$\mathrm{E}[u(\pi_{c0}^*) - u(\pi_{\ell0}^*)] > 0, \tag{14}$$

where $\pi_{c0}^* = f_0 q^* + [(1+\rho)\bar{k}/h^* + \delta](q - h^*) - c(q^*)$ and $\pi_{\ell0}^* = [f_0 + (1+\rho)\bar{k}/h^* + \delta]q^* - \bar{k} - c(q^*)$.

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[7]This is the case analyzed by Lien (2003) and Wong (2004a, 2004b), and Wong and Xu (2006) who restrict the correlation coefficient, ρ, to be zero, i.e., the futures price dynamics follows a random walk.

To examine the firm's optimal futures position, h^*, under the binding capital constraint, we let $L(\rho)$ be the left-hand side of Eq. (13) evaluated at $h^* = q^*$:

$$L(\rho) = -(1 + \rho)u'[f_0 q^* - c(q^*)] \int_{-\infty}^{\bar{k}/q^*} \varepsilon_1 g(\varepsilon_1) \, d\varepsilon_1$$

$$- \int_{\bar{k}/q^*}^{\infty} \mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\varepsilon_1 g(\varepsilon_1) \, d\varepsilon_1$$

$$+ \left\{\mathrm{E}\{u[(f_0 + \delta)q^* + \rho\bar{k} - c(q^*)]\} - u[f_0 q^* - c(q^*)]\right\}g(\bar{k}/q^*)\bar{k}/q^{*2}. \qquad (15)$$

Since ε_1 has a mean of zero, we can write Eq. (15) as

$$L(\rho) = \int_{\bar{k}/q^*}^{\infty} \left\{(1 + \rho)u'[f_0 q^* - c(q^*)] - \mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\right\}\varepsilon_1 g(\varepsilon_1) \, d\varepsilon_1$$

$$+ \left\{\mathrm{E}\{u[(f_0 + \delta)q^* + \rho\bar{k} - c(q^*)]\} - u[f_0 q^* - c(q^*)]\right\}g(\bar{k}/q^*)\bar{k}/q^{*2}. \qquad (16)$$

If $L(\rho) < (>) \, 0$, it follows from Eq. (13) and the second-order condition for program (3) that $h^* < (>) \, q^*$.

When there are multiple sources of uncertainty, it is well-known that the Arrow-Pratt theory of risk aversion is usually too weak to yield intuitively appealing results (Gollier, 2001). Kimball (1990, 1993) defines $u'''(\pi) \geq 0$ as prudence, which measures the propensity to prepare and forearm oneself under uncertainty, vis-à-vis risk aversion that is how much one dislikes uncertainty and would turn away from it if one could. As shown by Leland (1968), Drèze and Modigliani (1972), and Kimball (1990), prudence is both necessary and sufficient to induce precautionary saving. Moreover, prudence is implied by decreasing absolute risk aversion, which is instrumental in yielding many intuitive comparative statics under uncertainty (Gollier, 2001).

The following proposition characterizes the firm's optimal futures position, h^*, under the binding capital constraint, $k^* = \bar{k}$.

Proposition 5. *Given that the competitive firm is endowed with a finite amount of capital, $0 < \bar{k} < \infty$, and optimally devises its futures hedge program, (h^*, k^*), such that $k^* = \bar{k}$,*

the firm's optimal futures position, h^*, is an under-hedge, a full-hedge, or an over-hedge, depending on whether the autocorrelation coefficient, ρ, is less than, equal to, or greater than ρ^*, respectively, where $\rho^* > 0$ uniquely solves $L(\rho^*) = 0$, if the firm is prudent.

To see the intuition of Proposition 5, we refer to Eqs. (1) and (2). If the firm adopts a full-hedge, i.e., $h = q^*$, its random profit at $t = 2$ becomes

$$
\pi = \begin{cases}
f_0 q^* - c(q^*) & \text{if } \varepsilon_1 \leq \bar{k}/q^*, \\
(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*) & \text{if } \varepsilon_1 > \bar{k}/q^*.
\end{cases}
\tag{17}
$$

Eq. (17) implies that a full-hedge is not optimal due to the residual output price risk, $(\rho\varepsilon_1 + \delta)q^*$, that arises from the premature liquidation of the futures position at $t = 1$. According to Kimball (1990, 1993), the prudent firm is more sensitive to low realizations of its random profit at $t = 2$ than to high ones. If ρ is not too (is sufficiently) positive, i.e., $\rho < (>) \rho^*$, it is evident from Eq. (17) that the low realizations of the firm's random profit at $t = 2$ occur when the futures position is (is not) prematurely liquidated at $t = 1$. Thus, to avoid these realizations the prudent firm has incentives to short less (more) of the futures contracts, i.e., $h < (>) q^*$, so as to shrink (enlarge) the interval, $[\bar{k}/h, \infty)$, over which the premature liquidation of the futures position prevails at $t = 1$. The prudent firm as such optimally opts for an under-hedge (over-hedge), i.e., $h^* < (>) q^*$, when $\rho < (>) \rho^*$.

Proposition 5 characterizes the firm's optimal futures position only when the capital constraint is indeed binding, which is the case when condition (14) holds. The following proposition characterizes sufficient conditions under which condition (14) holds.

Proposition 6. *Given that the competitive firm is endowed with a finite amount of capital, $0 < \bar{k} < \infty$, and is prudent, the firm finds it optimal to put aside all of its capital stock, $\bar{k}$, for the futures hedge program, (h^*, k^*), if the autocorrelation coefficient, ρ, is non-positive.*

If the firm were endowed with an infinite amount of capital, we know from Proposition 2 that the firm would have chosen $k^* = \infty$ for all $\rho \leq 0$. Since $\bar{k}$ is in fact finite, it follows

that the firm optimally chooses $k^* = \bar{k}$ for all $\rho \leq 0$, as is shown in Proposition 6.

We now consider the case that the capital constraint is non-binding or just binding, i.e., $\lambda^* = 0$. From Proposition 6, we know that a necessary condition for this case is that $\rho > 0$. The Kuhn-Tucker conditions for program (3) under the non-binding or just-binding capital constraint become

$$\int_{-\infty}^{k^*/h^*} \mathrm{E}\{u'(\pi_c^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1$$

$$+ \int_{k^*/h^*}^{\infty} \mathrm{E}\{u'(\pi_\ell^*)[f_0 + (1+\rho)\varepsilon_1 + \delta - c'(q^*)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 = 0, \tag{18}$$

$$-\int_{-\infty}^{k^*/h^*} \mathrm{E}\{u'(\pi_c^*)[(1+\rho)\varepsilon_1 + \delta]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 - \int_{k^*/h^*}^{\infty} \mathrm{E}[u'(\pi_\ell^*)]\varepsilon_1 g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 = 0, \tag{19}$$

and

$$\mathrm{E}[u(\pi_{c0}^*) - u(\pi_{\ell0}^*)] = 0, \tag{20}$$

where $\pi_{c0}^* = f_0 q^* + [(1+\rho)k^*/h^* + \delta](q^* - h^*) - c(q^*)$ and $\pi_{\ell0}^* = [f_0 + (1+\rho)k^*/h^* + \delta]q^* - k^* - c(q^*)$. We derive the solution in the case that the firm's preferences exhibit constant absolute risk aversion in the following proposition.[8]

The following proposition characterizes the firm's optimal futures position, h^*, under the non-binding capital constraint, i.e., $k^* < \bar{k}$, or the just binding capital constraint, i.e., $k^* = \bar{k}$.

Proposition 7. *Given that the competitive firm is endowed with a finite amount of capital, $0 < \bar{k} < \infty$, and has constant absolute risk aversion, the firm's optimal futures position, h^*, is an over-hedge, i.e., $h^* > q$, and the autocorrelation coefficient, ρ, is greater than ρ^*, if the capital constraint is either non-binding or just binding.*

The intuition of Proposition 7 is similar to that of Proposition 4 and thus is omitted.

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[8]We do not consider increasing absolute risk aversion (IARA) because IARA is inconsistent with prudence.

6. Numerical examples

To gain more insights into the theoretical findings, we construct numerical examples to quantity the severity of the endogenous liquidity constraint, which is inversely gauged by the optimal liquidation threshold, k^*. We assume that the firm has a negative exponential utility function: $u(\pi) = -e^{-\gamma\pi}$, where $\gamma > 0$ is the constant Arrow-Pratt measure of absolute risk aversion. We further assume that ε_1 and δ are normally distributed with means of zero and variances of 0.01. For normalization, we set $q = f_0 = 1$ and $c(q) = 0$.

Table 1 report the firm's optimal futures position, h^*, for the case that the liquidation threshold, k, is exogenously set equal to the fixed capital endowment, $\bar{k}$. Setting $\gamma = 2$, we document the firm's optimal futures position, h^*, and the critical autocorrelation coefficient, ρ^*, for different values of k and ρ.

(Insert Table 1 here.)

As is evident from Table 1, $h^* < (>) 1$ when $\rho < (>) \rho^*$, in accord with Proposition 5. Table 1 also reveals that h^* moves further away from a full-hedge as $\bar{k}$ decreases. That is, when the exogenous liquidity constraint becomes more severe, the firm has to deviate more from full-hedging so as to better cope with the output price uncertainty and the liquidity risk simultaneously.

Table 2 reports the firm's optimal futures position, h^*, and the optimal liquidation threshold, k^*, when the capital constraint is not binding, i.e., $k^* < \bar{k}$. We document the firm's optimal futures hedge program, (h^*, k^*), for different values of γ and ρ.

(Insert Table 2 here.)

Table 2 shows that a full-hedge is optimal if ρ is small, or else an over-hedge is optimal, implying that an under-hedge is never used. It is also evident from Table 2 that k^* decreases as either ρ increases or γ decreases. That is, the firm is willing to commit itself to a more

aggressive (i.e., severe) liquidity constraint provided that premature liquidation is indeed ex-post profitable or that the firm is less risk averse and thus does not mind to take on excessive risk.

Table 3 reports the firm's optimal futures position, h^*, and the optimal liquidation threshold, k^*, when the capital constraint can be binding, i.e., $k^* = \bar{k}$. Setting $\gamma = 2$ and $\bar{k} = 0.2$, we document the firm's optimal futures hedge program, (h^*, k^*), for different values of ρ.

(Insert Table 3 here.)

It is evident from Table 3 that the capital constraint is binding, i.e., $k^* = \bar{k}$, for all $\rho < 0.05$, and is non-binding, i.e., $k^* < \bar{k}$, for all $\rho \geq 0.05$. Furthermore, in the case of non-binding capital constraint, the firm's optimal futures position, h^*, is an over-hedge, i.e., $h^* > q$, and the autocorrelation coefficient, ρ, exceeds the critical level, ρ^*, that is defined in Proposition 5. These results are consistent with Proposition 7.

7. Conclusion

In this paper, we have examined the optimal design of a futures hedge program by the competitive firm under output price uncertainty (Sandmo, 1971). The firm's futures hedge program consists of a futures position and an amount of capital earmarked for the program. The firm is subject to a capital constraint in that the earmarked capital cannot exceed the firm's capital endowment. Due to the capital constraint and the marking-to-market procedure of futures contracts, the firm faces endogenous liquidity risk. The futures price dynamics follows a first-order autoregression that includes a random walk as a special case.

When the futures prices are sufficiently positively correlated, we have shown that the capital constraint is non-binding. In this case, the optimal amount of capital earmarked to the futures hedge program is less than the firm's capital endowment. Furthermore,

the firm's optimal futures position is likely to be an over-hedge for reasonable preferences. When the futures prices are not too positively correlated, we have shown that the capital constraint is binding. In this case, the firm optimally puts aside all of its capital stock for the futures hedge program. Furthermore, the firm's optimal futures position is an under-hedge or an over-hedge, depending on whether the autocorrelation coefficient of the futures price dynamics is below or above a critical positive value, respectively.

Appendix A

The firm's ex-ante decision problem is to choose a futures position, h, so as to maximize the expected utility of its random profit at $t = 2$, EU:

$$\int_{-\infty}^{k/h} \mathrm{E}\Big\{ u\{f_0 q + [(1+\rho)\varepsilon_1 + \delta](q-h) - c(q)\} \Big\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1$$

$$+ \int_{k/h}^{\infty} \mathrm{E}\Big\{ u\{f_0 q + \varepsilon_1[(1+\rho)q - h] + \delta q - c(q)\} \Big\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 \tag{A.1}$$

if $h > 0$, and

$$\int_{-\infty}^{k/h} \mathrm{E}\Big\{ u\{f_0 q + \varepsilon_1[(1+\rho)q - h] + \delta q - c(q)\} \Big\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1$$

$$+ \int_{k/h}^{\infty} \mathrm{E}\Big\{ u\{f_0 q + [(1+\rho)\varepsilon_1 + \delta](q-h) - c(q)\} \Big\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 \tag{A.2}$$

if $h < 0$. In order to solve the firm's optimal futures position, h^*, we need to know which equation, Eq. (A.1) or Eq. (A.2), contains the solution.

Consider first the case that $h > 0$. Using Leibniz's rule to partially differentiate EU as defined in Eq. (A.1) with respect to h and evaluating the resulting derivative at $h \to 0^+$ yields

$$\lim_{h \to 0^+} \frac{\partial EU}{\partial h} = - \int_{-\infty}^{\infty} \mathrm{E}\Big\{ u'\{[f_0 + (1+\rho)\varepsilon_1 + \delta]q - c(q)\}[(1+\rho)\varepsilon_1 + \delta] \Big\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1. \tag{A.3}$$

Since ε_1 and δ has means of zero, the right-hand side of Eq. (A.3) is simply the negative of the covariance between $u'\{[f_0+(1+\rho)\varepsilon_1+\delta]q-c(q)\}$ and $(1+\rho)\varepsilon_1+\delta$ with respect to the joint probability density function of ε_1 and δ. Since $u''(\pi) < 0$, we have $\lim_{h\to 0^+}\partial EU/\partial h > 0$.

Now, consider the case that $h < 0$. Using Leibniz's rule to partially differentiate EU as defined in Eq. (A.2) with respect to h and evaluating the resulting derivative at $h \to 0^-$ yields

$$\lim_{h\to 0^-}\frac{\partial EU}{\partial h} = -\int_{-\infty}^{\infty} \mathrm{E}\Big\{u'\{[f_0 + (1 + \rho)\varepsilon_1 + \delta]q - c(q)\}[(1 + \rho)\varepsilon_1 + \delta]\Big\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1. \quad (A.4)$$

Inspection of Eqs. (A.3) and (A.4) reveals that $\lim_{h\to 0^+}\partial EU/\partial h = \lim_{h\to 0^-}\partial EU/\partial h > 0$. Since EU as defined in either Eq. (A.1) or Eq. (A.2) is strictly concave, the firm's optimal futures position, h^*, must be a short position, i.e., $h^* > 0$.

Appendix B

Proof of Proposition 1. Adding Eq. (10) to Eq. (9) yields

$$[f_0 - c'(q^0)]\int_{-\infty}^{\infty} \mathrm{E}[u'(\pi_c^0)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 = 0. \quad (B.1)$$

Since $u'(\pi) > 0$, Eq. (B.1) reduces to Eq. (11). If $h^0 = q^0$, the left-hand side of Eq. (10) becomes

$$-(1 + \rho)u'[f_0 q^0 - c(q^0)]\int_{-\infty}^{\infty} \varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 = 0, \quad (B.2)$$

since ε_1 and δ have means of zero. Inspection of Eqs. (10) and (B.2) reveals that $h^0 = q^0$ is indeed the optimal futures position.

Proof of Proposition 2. To facilitate the proof, we fix $h = q = q^0$ in program (3) to yield

$$\max_{k\geq 0}\ u[f_0 q^0 - c(q^0)]\int_{-\infty}^{k/q^0} g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$+ \int_{k/q^0}^{\infty} \mathrm{E}\{u[(f_0 + \rho\varepsilon_1 + \delta)q^0 - c(q^0)]\}g(\varepsilon_1) \, \mathrm{d}\varepsilon_1. \tag{B.3}$$

The Kuhn-Tucker condition for program (B.3) is given by

$$\left\{ u[f_0 q^0 - c(q^0)] - \mathrm{E}\{u[(f_0 + \delta)q^0 + \rho k^0 - c(q^0)]\} \right\} g(k^0/q^0)/q^0 \geq 0, \tag{B.4}$$

where k^0 is the optimal liquidity threshold when $h = q = q^0$. Should $k^0 < \infty$, condition (B.4) holds with equality.

If $\rho \leq 0$, it follows from $u''(\pi) < 0$, $\mathrm{E}(\delta) = 0$, and Jensen's inequality that $u[f_0 q^0 - c(q^0)] \geq u[f_0 q^0 + \rho k^0 - c(q^0)] > \mathrm{E}\{u[(f_0 + \delta)q^0 + \rho k^0 - c(q^0)]\}$, and thus $k^0 = \infty$ by condition (B.4). Since Proposition 1 implies that $h^* = q^* = q^0$ if $k^* = \infty$, it must be the case that $h^* = q^* = q^0$ and $k^* = \infty$ if $k^0 = \infty$. On the other hand, if $\rho > 0$, it is evident that $\mathrm{E}\{u[(f_0 + \delta)q^0 + \rho k - c(q^0)]\}$ is increasing in k. When $k = 0$, it follows from $u''(\pi) < 0$, $\mathrm{E}(\delta) = 0$, and Jensen's inequality that $\mathrm{E}\{u[(f_0 + \delta)q^0 - c(q^0)]\} < u[f_0 q^0 - c(q^0)]$. Also, for k sufficiently large, it must be the case that $\mathrm{E}\{u[(f_0 + \delta)q^0 + \rho k - c(q^0)]\} > u[f_0 q^0 - c(q^0)]$. Thus, there exists a unique point, $k^0 \in (0, \infty)$, such that condition (B.4) holds with equality. Suppose that $k^* = \infty$ but $k^0 < \infty$. It then follows from Proposition 1 that $h^* = q^* = q^0$, which would imply that $k^0 = k^* = \infty$, a contradiction to $k^0 < \infty$.

Proof of Proposition 3. When $\rho \leq 0$, Proposition 2 implies that $k^* = \infty$. It then follows from Proposition 1 that $h^* = q^* = q^0$.

Proof of Proposition 4. To facilitate the proof, we reformulate the firm's ex-ante decision problem as a two-stage optimization problem with q fixed at q^*. In the first stage, the firm chooses its optimal liquidation threshold, $k(h)$, for a given futures position, h:

$$k(h) = \arg\max_{k \geq 0} \int_{-\infty}^{k/h} \mathrm{E}[u(\pi_c)]g(\varepsilon_1) \, \mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}[u(\pi_\ell)]g(\varepsilon_1) \, \mathrm{d}\varepsilon_1, \tag{B.5}$$

where π_ℓ and π_c are given in Eqs. (1) and (2) with $q = q^*$, respectively. In the second stage, the firm chooses its optimal futures position, h^*, taking the liquidation threshold, $k(h)$, as

given by Eq. (B.5):

$$\max_{h} F(h) = \int_{-\infty}^{k(h)/h} E[u(\pi_c)]g(\varepsilon_1) \, d\varepsilon_1 + \int_{k(h)/h}^{\infty} E[u(\pi_\ell)]g(\varepsilon_1) \, d\varepsilon_1, \tag{B.6}$$

where π_ℓ and π_c are given in Eqs. (1) and (2) with $q = q^*$ and $k = k(h)$, respectively. The complete solution is thus given by h^* and $k^* = k(h^*)$.

Differentiating $F(h)$ in Eq. (B.6) with respect to h, using the envelope theorem, and evaluating the resulting derivative at $h = q^*$ yields

$$F'(q^*) = -(1+\rho)u'[f_0 q^* - c(q^*)] \int_{-\infty}^{k(q^*)/q^*} \varepsilon_1 g(\varepsilon_1) \, d\varepsilon_1$$

$$- \int_{k(q^*)/q^*}^{\infty} E\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\varepsilon_1 g(\varepsilon_1) \, d\varepsilon_1, \tag{B.7}$$

where $k(q^*)$ solves

$$u[f_0 q^* - c(q^*)] = E\{u[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}. \tag{B.8}$$

It is evident from Eq. (B.8) that $\rho k(q^*)$ is equal to the risk premium of the zero-mean risk, δq^*, in the usual Arrow-Pratt sense.

Rewrite Eq. (B.8) as

$$u[f_0 q^* - c(q^*) + m] = E\{u[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*) + m]\}, \tag{B.9}$$

where m can be interpreted as endowed wealth that takes on an initial value of zero. Differentiating Eq. (B.9) with respect to m and evaluating the resulting derivative at $m = 0$ yields

$$\left.\frac{\partial \rho k(q^*)}{\partial m}\right|_{m=0} = \frac{u'[f_0 q^* - c(q^*)] - E\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}}{E\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}}. \tag{B.10}$$

If $u(\pi)$ satisfies decreasing, constant, or increasing absolute risk aversion (DARA, CARA, or IARA), $\partial \rho k(q^*)/\partial m$ is negative, zero, or positive, respectively. Using the fact that ε_1 has a mean of zero, Eq. (B.7) can be written as

$$F'(q^*) = \int_{k(q^*)/q^*}^{\infty} \left\{ (1+\rho)u'[f_0 q^* - c(q^*)] \right.$$

$$-\mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\bigg\}\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1. \tag{B.11}$$

If $u(\pi)$ satisfies CARA (IARA), Eq. (B.10) implies that

$$u'[f_0 q^* - c(q^*)] = (>) \, \mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}. \tag{B.12}$$

Since $\rho > 0$, Eq. (B.22) and risk aversion imply that $(1 + \rho)u'[f_0 q^* - c(q^*)] > \mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\} > \mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}$ for all $\varepsilon_1 > k(q^*)/q^*$. It then follows from Eq. (B.11) that $F'(q^*) > 0$ and thus $h^* > q^*$ if $u(\pi)$ satisfies either CARA or IARA.

Proof of Proposition 5. Differentiating $L(\rho)$ with respect to ρ yields

$$L'(\rho) = \int_{\bar{k}/q^*}^{\infty} \bigg\{ u'[f_0 q^* - c(q^*)] - \mathrm{E}\{u''[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\varepsilon_1 q \bigg\}\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$+\mathrm{E}\{u'[(f_0 + \delta)q^* + \rho\bar{k} - c(q^*)]\}g(\bar{k}/q^*)\bar{k}^2/q^{*2}. \tag{B.13}$$

Since $u'(\pi) > 0$ and $u''(\pi) < 0$, Eq. (B.13) implies that $L'(\rho) > 0$. Evaluating Eq. (16) at $\rho = 0$ yields

$$L(0) = \bigg\{ u'[f_0 q^* - c(q^*)] - \mathrm{E}\{u'[(f_0 + \delta)q^* - c(q^*)]\} \bigg\} \int_{\bar{k}/q^*}^{\infty} \varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$+\bigg\{ \mathrm{E}\{u[(f_0 + \delta)q^* - c(q^*)]\} - u[f_0 q^* - c(q^*)] \bigg\}g(\bar{k}/q^*)\bar{k}/q^{*2}. \tag{B.14}$$

Since $u''(\pi) < 0$ and $\mathrm{E}(\delta) = 0$, Jensen's inequality implies that $\mathrm{E}\{u[(f_0 + \delta)q^* - c(q^*)]\} < u[f_0 q^* - c(q^*)]$. The second term on the right-hand side of Eq. (B.14) is negative. Since $u'''(\pi) \geq 0$, it follows from $\mathrm{E}(\delta) = 0$ and Jensen's inequality that $\mathrm{E}\{u'[(f_0 + \delta)q^* - c(q^*)]\} \geq u'[f_0 q^* - c(q^*)]$. The first term on the right-hand side of Eq. (B.14) is non-positive and thus $L(0) < 0$. Now, consider the case that ρ is sufficiently large such that $(1 + \rho)u'[f_0 q^* - c(q^*)] > \mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}$ for all $\varepsilon_1 > 0$ and $u[f_0 q^* - c(q^*)] < \mathrm{E}\{u[(f_0 + \delta)q^* + \rho\bar{k} - c(q^*)]\}$. Thus, for ρ sufficiently large, it follows from Eq. (16) that $L(\rho) > 0$. Since $L(0) < 0$, $L(\rho) > 0$ for ρ sufficiently large, and $L'(\rho) > 0$, there must exist a unique point, $\rho^* > 0$,

that solves $L(\rho^*) = 0$. Thus, for all $\rho < (>) \rho^*$, we have $L(\rho) < (>) 0$. It then follows from Eq. (13) and the second-order condition for program (3) that $h^* < (>) q^*$ for all $\rho < (>) \rho^*$.

Proof of Proposition 6. Let $\Phi(\pi_{\ell 0}^*)$ and $\Psi(\pi_{c0}^*)$ be the cumulative distribution functions (CDFs) of $\pi_{\ell 0}^*$ and π_{c0}^* defined in condition (14), respectively, and let

$$T(x) = \int_{-\infty}^{x} [\Phi(y) - \Psi(y)] \, \mathrm{d}y. \tag{B.15}$$

Using Eq. (B.15), we can write the left-hand side of condition (14) as

$$\mathrm{E}[u(\pi_{\ell 0}^*)] - \mathrm{E}[u(\pi_{c0}^*)] = \int_{-\infty}^{\infty} u(x) \, \mathrm{d}[\Phi(x) - \Psi(x)] = \int_{-\infty}^{\infty} u''(x) T(x) \, \mathrm{d}x, \tag{B.16}$$

where the second equality follows from $u'(\infty) = 0$ and integration by parts. In light of Eq. (B.16), condition (14) holds if $\Phi(x)$ is either a second-order stochastic dominance shift or a mean-preserving-spread shift of $\Psi(x)$.

Note that $T(-\infty) = 0$ and

$$T(\infty) = \int_{-\infty}^{\infty} [\Phi(x) - \Psi(x)] \, \mathrm{d}x = \int_{-\infty}^{\infty} x \, \mathrm{d}\Psi(x) - \int_{-\infty}^{\infty} x \, \mathrm{d}\Phi(x) = -\rho \bar{k}, \tag{B.17}$$

where the second equality follows from integration by parts. Since $\rho \leq 0$, Eq. (B.17) implies that $T(\infty) \geq 0$, where the equality holds only when $\rho = 0$. We can write

$$\pi_{\ell 0}^* = \pi_{c0}^* + \rho \bar{k} + \delta h^*$$

$$= \pi_{c0}^* + \left[\pi_{c0}^* - f_0 q^* - \bar{k}\left(\frac{q^* - h^*}{h^*}\right) + c(q^*) \right]\left(\frac{h^*}{q^* - h^*}\right). \tag{B.18}$$

Using the change-of-variable technique (Hogg and Craig, 1989) and Eq. (B.18), we have $\Psi(\pi_{c0}^*) = \Phi\{\pi_{c0}^* + [\pi_{c0}^* - f_0 q^* - \bar{k}(q^* - h^*)/h^* + c(q^*)]h^*/(q^* - h^*)\}$. Differentiating $T(x)$ in Eq. (B.15) with respect to x and using Leibniz's rule yields $T'(x) = \Phi(x) - \Psi(x)$. It follows from $\Psi(x) = \Phi\{x + [x - f_0 q^* - \bar{k}(q^* - h^*)/h^* + c(q^*)]h^*/(q^* - h^*)\}$ and $h^* < q^*$ that $\Phi(x) - \Psi(x) > (<) 0$ if $x < (>) f_0 q^* + \bar{k}(q^* - h^*)/h^* - c(q^*)$. Hence, $T(x)$ is strictly

increasing for all $x < f_0 q^* + \bar{k}(q^* - h^*)/h^* - c(q^*)$ and strictly decreasing for all $x >$
$f_0 q^* + \bar{k}(q^* - h^*)/h^* - c(q^*)$. Since $T(-\infty) = 0$, $T(\infty) \geq 0$, and $T(x)$ is first increasing and
then decreasing in x, we have $T(x) > 0$ for all x. In other words, $\Phi(x)$ is a second-order
stochastic dominance shift of $\Psi(x)$ for all $\rho < 0$ and is a mean-preserving-spread shift of
$\Psi(x)$ when $\rho = 0$. Thus, for all $\rho \leq 0$, Eq. (B.16) implies that $\mathrm{E}[u(\pi_{c0}^*)] > \mathrm{E}[u(\pi_{\ell 0}^*)]$ given
risk aversion.

Proof of Proposition 7.   To facilitate the proof, the firm's ex-ante decision problem is
formulated as a two-stage optimization problem. In the first stage, the firm chooses its
optimal liquidation threshold, $k(h)$, for a given short futures position, h:

$$k(h) = \arg\max_{k \geq 0} \int_{-\infty}^{k/h} \mathrm{E}[u(\pi_c)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}[u(\pi_\ell)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1, \qquad (\mathrm{B.19})$$

where π_ℓ and π_c are given in Eqs. (1) and (2) with $q = q^*$, respectively. In the second stage,
the firm chooses its optimal futures position, h^*, taking the liquidation threshold, $k(h)$, as
given by Eq. (B.19):

$$\max_h \ G(h) = \int_{-\infty}^{k(h)/h} \mathrm{E}[u(\pi_c)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 + \int_{k(h)/h}^{\infty} \mathrm{E}[u(\pi_\ell)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1, \qquad (\mathrm{B.20})$$

where π_ℓ and π_c are given in Eqs. (1) and (2) with $q = q^*$ and $k = k(h)$, respectively. The
complete solution is thus given by h^* and $k^* = k(h^*)$, which also solves Eqs. (19) and (20).

Differentiating $G(h)$ in Eq. (B.20) with respect to h, using the envelope theorem, and
evaluating the resulting derivative at $h = q^*$ yields

$$G'(q^*) = -(1+\rho)u'[f_0 q^* - c(q^*)]\int_{-\infty}^{k(q^*)/q^*} \varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$- \int_{k(q^*)/q^*}^{\infty} \mathrm{E}\{u'[(f_0 + \rho\varepsilon_1 + \delta)q^* - c(q^*)]\}\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1, \qquad (\mathrm{B.21})$$

where $k(q^*)$ solves

$$u[f_0 q^* - c(q^*)] = \mathrm{E}\{u[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}. \qquad (\mathrm{B.22})$$

It is evident from Eq. (B.22) that $\rho k(q^*)$ is equal to the risk premium of the zero-mean risk, δq^*, in the usual Arrow-Pratt sense.

Rewrite Eq. (B.22) as

$$u[f_0 q^* - c(q^*) + m] = \mathrm{E}\{u[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*) + m]\}, \tag{B.23}$$

where m can be interpreted as endowed wealth that takes on an initial value of zero. Differentiating Eq. (B.23) with respect to m and evaluating the resulting derivative at $m = 0$ yields

$$\left.\frac{\partial \rho k(q^*)}{\partial m}\right|_{m=0} = \frac{u'[f_0 q^* - c(q^*)] - \mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}}{\mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}}. \tag{B.24}$$

Since $u(\pi)$ satisfies CARA, Eq. (B.24) implies that

$$u'[f_0 q^* - c(q^*)] = \mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\}. \tag{B.25}$$

Using the fact that ε_1 has a mean of zero, we can write Eq. (B.21) as

$$G'(q^*) = \int_{k(q^*)/q^*}^{\infty} \left\{ (1 + \rho)u'[f_0 q^* - c(q^*)] \right.$$

$$\left. - \mathrm{E}\{u'[(f_0 + \rho \varepsilon_1 + \delta)q^* - c(q^*)]\} \right\} \varepsilon_1 g(\varepsilon_1) \, \mathrm{d}\varepsilon_1. \tag{B.26}$$

Since $\rho > 0$, Eq. (B.25) and risk aversion imply that $(1 + \rho)u'[f_0 q^* - c(q^*)] > \mathrm{E}\{u'[(f_0 + \delta)q^* + \rho k(q^*) - c(q^*)]\} > \mathrm{E}\{u'[(f_0 + \rho \varepsilon_1 + \delta)q^* - c(q^*)]\}$ for all $\varepsilon_1 > k(q^*)/q^*$. It then follows from Eq. (B.26) that $G'(q^*) > 0$ and thus $h^* > q^*$ if $u(\pi)$ satisfies CARA.

Suppose that there is a point, $\rho_1 \in (0, \rho^*]$, at which the capital constraint is either non-binding or just binding. Since $k^* = \bar{k}$ for all $\rho \leq 0$ according to Proposition 2, continuity implies that there must exist a point, $\rho_2 \in (0, \rho_1]$, such that the capital constraint is just binding, i.e., $k^* = \bar{k}$. Since $\rho_2 \leq \rho^*$, Proposition 1 implies that $h^* \leq q$ at ρ_2, a contradiction to our conclusion that $h^* > q$ if the capital constraint is either non-binding or just binding. Hence, our supposition is wrong so that $\rho > \rho^*$ whenever the capital constraint is either non-binding or just binding.

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Table 1

Optimal futures positions under exogenous liquidity constraints

	$\bar{k} = 0.05$		$\bar{k} = 0.1$		$\bar{k} = 0.15$		$\bar{k} = 0.2$	
	ρ^*	h^*	ρ^*	h^*	ρ^*	h^*	ρ^*	h^*
$\rho = 0.01$		0.9381		0.9337		0.9566		0.9787
$\rho = 0.03$		0.9728		0.9659		0.9801		0.9906
	0.0460	1.0000	0.0514	1.0000	0.0462	1.0000	0.0395	1.0000
$\rho = 0.07$		1.0391		1.0286		1.0287		1.0225
$\rho = 0.09$		1.0707		1.0592		1.0538		1.0383

Notes: The competitive firm has a negative exponential utility function: $u(\pi) = -e^{-2\pi}$. The underlying random variables, ε_1 and δ, are normally distributed with means of zero and variances of 0.01. Both the level of output, q, and the initial futures price, f_0, are normalized to unity. The liquidation threshold, k, is exogenously set equal to the firm's fixed amount of capital, $\bar{k}$. This table reports the optimal futures position, h^*, and the critical autocorrelation coefficient, ρ^*, for different values of the exogenous liquidity constraint, $\bar{k}$, and the autocorrelation coefficient, ρ.

Table 2
Optimal futures hedge programs under non-binding capital constraints

	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$		$\gamma = 4$	
	h^*	k^*	h^*	k^*	h^*	k^*	h^*	k^*
$\rho = 0.01$	1.0000	0.5000	1.0000	0.9884	1.0000	1.4852	1.0000	1.4998
$\rho = 0.02$	1.0000	0.2494	1.0000	0.5000	1.0000	0.7481	1.0000	0.9901
$\rho = 0.05$	1.0726	0.1000	1.0070	0.1999	1.0000	0.2970	1.0000	0.3990
$\rho = 0.1$	1.2313	0.0473	1.0738	0.0997	1.0188	0.1485	1.0073	0.2000
$\rho = 0.2$	1.5103	0.0185	1.2385	0.0472	1.1349	0.0737	1.1031	0.0990
$\rho = 0.5$	2.1670	0.0036	1.6292	0.0121	1.4354	0.0243	1.3294	0.0357

Notes: The competitive firm has a negative exponential utility function: $u(\pi) = -e^{-\gamma\pi}$, where γ is a positive constant. The underlying random variables, ε_1 and δ, are normally distributed with means of zero and variances of 0.01. Both the level of output, q, and the initial futures price, f_0, are normalized to unity. The capital constraint is assumed to be non-binding, i.e., $k^* < \bar{k}$. This table reports the optimal futures position, h^*, and the optimal liquidation threshold, k^*, for different values of the risk aversion coefficient, γ, and the autocorrelation coefficient, ρ.

Table 3
Optimal futures hedge programs

ρ	0.01	0.03	0.0395	0.05	0.1	0.2	0.5
h^*	0.9787	0.9906	1.0000	1.0070	1.0738	1.2385	1.6292
k^*	0.2	0.2	0.2	0.1999	0.0997	0.0472	0.0121

Notes: The competitive firm has a negative exponential utility function: $u(\pi) = -e^{-2\pi}$ and is endowed with a fixed amount of capital, $\bar{k}$, set equal to 0.2. The underlying random variables, ε_1 and δ, are normally distributed with means of zero and variances of 0.01. Both the level of output, q, and the initial futures price, f_0, are normalized to unity. The critical value, ρ^*, as defined in Proposition 5 is 0.0395. This table reports the optimal futures position, h^*, and the optimal liquidation threshold, k^*, for different values of the autocorrelation coefficient, ρ.