Cross-Hedging of Exchange Rate Risks:
A Note

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This note studies the optimal production and hedging decisions of a competitive international firm that exports to two foreign countries. The firm as such faces multiple sources of exchange rate uncertainty. Cross-hedging is plausible in that one of these two foreign countries has a currency forward market. We show that the separation theorem holds in that the firm's production decision depends neither on its risk attitude nor on the underlying uncertainty. We further show that the firm's optimal forward position is an over-hedge, a full-hedge, or an under-hedge, depending on whether the two random exchange rates are strongly positively correlated, uncorrelated, or negatively correlated, respectively.

1. Introduction

Financial markets in the real world are far from complete. For example, not all currencies have forward markets (see Eiteman, Stonehill, and Moffett, 1998). This is especially prominent in less developed countries (LDCs) where capital markets are embryonic and foreign exchange markets are heavily controlled. Even if currency forward contracts are available in some LDCs, they deem to be forward-cover insurance schemes that are not governed by market forces (see Jacque, 1996). International firms that expose to currencies of these countries thus have to rely on forward contracts on related currencies to indirectly hedge against their exchange rate risk exposure. Such an exchange rate risk management technique is referred to as "cross-hedging" (see, e.g., Anderson and Danthine, 1981; Broll, Wong, and Zilcha; 1999; and Chang and Wong, 2003).

The purpose of this note is to provide theoretical insights into the optimal cross-hedging strategies of international firms. To this end, we consider a risk-averse competitive firm that exports its output to two foreign countries. Only one of these two foreign countries has a currency forward market to which the firm has access. We show that the celebrated separation theorem (see, e.g., Katz and Paroush, 1979; Kawai and Zilcha, 1986; Broll and Zilcha, 1992;

and Broll, Wong, and Zilcha, 1999) holds. Specifically, the firm's production decision is independent of its risk attitude and of the underlying exchange rate uncertainty. We further show that the firm's optimal forward position depends on the bivariate dependence of the random exchange rates. To derive concrete results, we propose the concepts of strong correlation. We show that over-hedging, full-hedging, or under-hedging is optimal, depending on whether the two random exchange rates are strongly positively correlated, uncorrelated, or negatively uncorrelated, respectively.

The rest of this note is organized as follows. The next section develops the model of an international firm facing exchange rate uncertainty and cross-hedging opportunities. Section 3 characterizes the firm's optimal output and cross-hedging strategy. The final section offers some concluding remarks.

2. The model

We consider a competitive international firm that produces a single output, x, according to a cost function, $c(x)$, where $c'(0) \geq 0$, $c'(x) > 0$, and $c''(x) > 0$. The firm exports its entire output to two foreign countries (indexed by $i = 1$ and 2). Let x_i and P_i be the amount of exports and the per-unit selling price in country i, where $i = 1$ and 2 and $x_1 + x_2 = x$. Exchange rate uncertainty comes from two sources, $\tilde{e}_1$ and $\tilde{e}_2$, that denote the random exchange rates expressed in units of the domestic currency per unit of country 1's currency and country 2's currency, respectively.[1] Cross-hedging is modeled by allowing the firm to trade infinitely divisible forward contracts between country 1's currency and the domestic currency at the forward rate, e_1^f. There are no direct hedging instruments for the random exchange rate, $\tilde{e}_2$.

The firm's profits, denominated in the domestic currency, are given by

$$\tilde{\Pi} = \tilde{e}_1 P_1 x_1 + \tilde{e}_2 P_2 x_2 - c(x) + (e_1^f - \tilde{e}_1)h, \tag{1}$$

where h is the number of the forward contracts sold (purchased if negative). The firm is risk averse and possesses a von Neumann-Morgenstern utility function, $U(\Pi)$, defined over its domestic currency profits, Π, with $U'(\Pi) > 0$ and $U''(\Pi) < 0$. The firm's *ex ante* decision problem is to choose its output,

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[1] A tilde ($\sim$) denotes a random variable.

x, exports, x_1 and x_2, and forward position, h, so as to maximize the expected utility of its domestic currency profits:

$$\max_{x,x_1,x_2,h} \ E[U(\tilde{\Pi})] \ \ \text{s.t.} \ \ x_1 + x_2 = x, \tag{2}$$

where $E(\cdot)$ is the expectation operator.

3. Optimal production and hedging decisions

Assuming an interior optimal solution and substituting $x_1 = x - x_2$, the first-order conditions for program (2) are given by[2]

$$E\{U'(\tilde{\Pi}^*)[\tilde{e}_1 P_1 - c'(x^*)]\} = 0, \tag{3}$$

$$E[U'(\tilde{\Pi}^*)(\tilde{e}_2 P_2 - \tilde{e}_1 P_1)] = 0, \tag{4}$$

$$E[U'(\tilde{\Pi}^*)(e_1^f - \tilde{e}_1)] = 0, \tag{5}$$

where an asterisk (*) indicates an optimal level. The second-order conditions for program (2) are satisfied given the assumed properties of $U(\Pi)$ and $c(x)$.

Proposition 1. *If the competitive international firm is allowed to trade the forward contracts between country 1's currency and the domestic currency, then the firm's optimal output, x^*, solves*

$$c'(x^*) = e_1^f P_1. \tag{6}$$

Proof. Multiplying P_1 to equation (5) and adding the resulting equation to equation (3) yields

$$E[U'(\tilde{\Pi}^*)][e_1^f P_1 - c'(x^*)] = 0. \tag{7}$$

Since $U'(\Pi) > 0$, equation (7) reduces to equation (6).

Proposition 1 is simply the celebrated separation theorem (see, e.g., Katz and Paroush, 1979; Kawai and Zilcha, 1986; Broll and Zilcha, 1992; and

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[2]A necessary condition for an interior optimal solution is that $E(\tilde{e}_2)P_2 > e_1^f P_1$.

Broll, Wong, and Zilcha, 1999), which states that a firm's production decision depends neither on the risk attitude of the firm nor on the underlying uncertainty should the firm have access to a forward market. To see the intuition of Proposition 1, we recast equation (1) as

$$\tilde{\Pi} = e_1^f P_1(x - x_2) - c(x) + \tilde{e}_2 P_2 x_2 + (e_1^f - \tilde{e}_1)[h - P_1(x - x_2)]. \qquad (8)$$

Inspection of equation (8) reveals that the firm could have completely eliminated its exchange rate risk exposure had it chosen $x_2 = 0$ and $h = P_1(x - x_2)$ within its own discretion. Alternatively put, the degree of exchange rate risk exposure to be assumed by the firm should be totally unrelated to its production decision. The optimal output, x^*, is then chosen to maximize $e_1^f P_1(x - x_2) - c(x)$, which yields equation (6).

To examine the firm's optimal forward position, h^*, we write equation (5) as[3]

$$E[U'(\tilde{\Pi}^*)][e_1^f - E(\tilde{e}_1)] - Cov[U'(\tilde{\Pi}^*), \tilde{e}_1] = 0, \qquad (9)$$

where $Cov(\cdot, \cdot)$ is the covariance operator. Evaluating the left-hand side of equation (9) at $h^* = P_1 x_1^*$ yields

$$E\{U'[e_1^f P_1 x_1^* + \tilde{e}_2 P_2 x_2^* - c(x^*)]\}[e_1^f - E(\tilde{e}_1)]$$

$$-Cov\{U'[e_1^f P_1 x_1^* + \tilde{e}_2 P_2 x_2^* - c(x^*)], \tilde{e}_1\}. \qquad (10)$$

If the above expression is positive, zero, or negative, equation (9) and the strict concavity of $E[U'(\tilde{\Pi})]$ imply that h^* is greater than, equal to, or less than $P_1 x_1^*$, respectively.

Without imposing some concepts of bivariate dependence upon $\tilde{e}_1$ and $\tilde{e}_2$, it is impossible to determine the sign of expression (10). As such, we offer the following definition.

Definition. *The random variable, $\tilde{x}$, is said to be strongly positively correlated, uncorrelated, or negatively correlated to the random variable, $\tilde{y}$, if, and only if, $Cov[\tilde{x}, f(\tilde{y})]$ is positive, zero, or negative, respectively, for all strictly increasing functions, $f(\cdot)$.*

This definition is motivated by similarly ordered random variables in Hardy, Littlewood, and Pólya (1937) and Ingersoll (1987). An example

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[3] For any two random variables, $\tilde{x}$ and $\tilde{y}$, we have $Cov(\tilde{x}, \tilde{y}) = E(\tilde{x}\tilde{y}) - E(\tilde{x})E(\tilde{y})$.

of strongly correlated random variables is the linear specification: $\tilde{e}_2 = \alpha + \beta\tilde{e}_1 + \tilde{\epsilon}$, where α and β are scalars, and $\tilde{\epsilon}$ is a zero-mean random variable independent of $\tilde{e}_1$. This linear specification is widely used in the hedging literature (see, e.g., Benninga, Eldor, and Zilcha, 1983; Briys, Crouhy, and Schlesinger, 1993; and Broll, Wong, and Zilcha, 1999).

Proposition 2. *Given that the competitive international firm is allowed to trade the forward contracts between country 1's currency and the domestic currency. If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly uncorrelated, then the firm's optimal forward position, h^*, is greater than, equal to, or less than P_1x^*, depending on whether e_1^f is greater than, equal to, or less than $E(\tilde{e}_1)$, respectively. If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly positively (negatively) correlated, then the firm's optimal forward position, h^*, is greater (less) than P_1x^* when $e_1^f \geq (\leq) E(\tilde{e}_1)$.*

Proof. If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly uncorrelated, then the covariance term of expression (10) vanishes. Thus, expression (10) is positive, zero, or negative, depending on whether e_1^f is greater than, equal to, or less than $E(\tilde{e}_1)$, which implies that h^* is greater than, equal to, or less than $P_1x_1^*$, respectively.

If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly positively (negatively) correlated, then the covariance term of expression (10) is positive (negative). Thus, expression (10) is positive (negative) when $e_1^f \geq (\leq) E(\tilde{e}_1)$ so that $h^* > (<) P_1x^*$.

The intuition of Proposition 2 is as follows. Taking variance on both sides of equation (1), we have

$$Var(\tilde{\Pi}) = Var(\tilde{e}_1)(P_1x_1 - h)^2 + Var(\tilde{e}_2)P_2^2x_2^2$$

$$+2Cov(\tilde{e}_1, \tilde{e}_2)(P_1x_1 - h)P_2x_2, \tag{11}$$

where $Var(\cdot)$ is the variance operator. Partially differentiating equation (11) with respect to h and evaluating the resulting derivative at $h = P_1x_1$ yields

$$\frac{\partial}{\partial h}Var(\tilde{\Pi})\Big|_{h=P_1x_1} = -2Cov(\tilde{e}_1, \tilde{e}_2)P_2x_2. \tag{12}$$

If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly positively (negatively) correlated, we have $Cov(\tilde{e}_1, \tilde{e}_2) > (<) 0$. To reduce the variability of its domestic currency profits, the firm finds it optimal to set $h > (<) P_1x_1$ according to equation (12).

When $e_1^f > (<) E(\tilde{e}_1)$, there is a speculative motive that induces the firm to sell (purchase) the forward contracts. Thus, the over-hedging (under-hedging) incentive for risk minimization is reinforced by the speculative motive when $e_1^f \geq (\leq) E(\tilde{e}_1)$.

If $\tilde{e}_1$ and $\tilde{e}_2$ are strongly uncorrelated, we have $Cov(\tilde{e}_1, \tilde{e}_2) = 0$. Thus, equation (12) implies that $h = P_1 x_1$ minimizes the variability of the firm's domestic currency profits. The firm deviates from this full-hedge only when $e_1^f \neq E(\tilde{e}_1)$. If $e_1^f > (<) E(\tilde{e}_1)$, the speculative motive induces the firm to sell (purchase) the forward contracts, thereby making over-hedging (under-hedging) optimal.

4. Concluding remarks

In this note, we have examined the optimal production and hedging decisions of a competitive international firm facing multiple sources of exchange rate uncertainty. The firm exports its output to two foreign countries, only one of which has a currency forward market. We have shown that the celebrated separation theorem holds in that the firm's production decision does not depend on its risk attitude and the underlying exchange rate uncertainty. Furthermore, we have shown that the firm's optimal forward position is an over-hedge, a full-hedge, or an under-hedge, depending on whether the two random exchange rates are strongly positively correlated, uncorrelated, or negatively correlated, respectively.

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