On the Pricing and Hedging

of Discrete Dynamic Guaranteed Funds

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Abstract

This paper investigates the possibility of hedging discrete stochastic jumps and their tradeoffs in guaranteed funds under discrete dynamic hedging. Since a guaranteed fund price process is composed of diffusion and jump process, its expected rate of return is above the risk-free rate of interest. When delta dynamic hedging occurs at discrete instants, the rate differential will be manifested in non-zero expected hedging errors. We employ the dynamic guaranteed fund as our example, whose exotic fund structure excludes the possibility of static hedge. We derive a pricing model and develop hedging formulas for discrete dynamic guaranteed funds. We show our discrete-time delta hedging formulas induce smaller hedging errors than those based on applying the continuous-time hedging formula of Gerber and Pafumi (2000) at discrete instants. Nevertheless, this discrete-time model still incurs significant negative expected hedging errors induced partly by the guarantee jumps. We introduce a gamma-adjusted delta hedging strategy. The simulation results indicate that the strategy can effectively improve the discrete hedging performance of dynamic guaranteed funds.

1. Introduction

The increasing competition from peer financial institutions, the privatization of government social security programs[1], and the need for protection by investors against downside risk have led to the inclusion of financial guarantees and other embedded options in many contemporary insurance products. Equity-linked insurance products that expose their investors or policy holders to investment risk are offered worldwide. These products usually protect their investors by a fixed benefit, or a multi-period minimum rate of return. The performance of these insurance products is typically linked to some reference portfolios, e.g., a stock market index. Brennann and Schwartz (1976) value a single-period maturity guaranteed equity-linked contract as an insurance contract with an embedded put option. Boyle and Schwartz (1977) determine an optimal investment policy between investing in the reference portfolio and a riskless reserve for a fund issuer to hedge against the investment risk of these guarantees.

Investment products of multi-period guarantees can be found in guaranteed investment contracts (GIC) sold by investment banks (see Miltersen and Persson (1999)). Many life insurance companies also offer contracts which guarantee the policy holders a fixed minimum annual percentage return. Grosen and Jorgensen (1997) investigate the value of a life insurance policy which contains an explicit guarantee that ensures the investor, upon surrendering, a certain minimum return during some specified period. In essence, these insurance contracts embed in them some forms of American-style early-exercisable options which require active monitoring of the value of the underlying insurance product by the policy holders in order to make informed early-exercise and contract rollover (Pedersen and Shiu (1994)) decisions.

In a series of papers, Gerber and Shiu (1999), Gerber and Pafumi (2000), Imai and Boyle (2001), and Gerber and Shiu (2003) introduce a dynamic guaranteed fund featured by automatic multi-period reset guarantees. The guarantees are dynamic because the fund price is upgraded through cash injection by the issuer whenever it falls below a certain threshold during its life. Thus, the fund provides a floor protection for an equity-index linked portfolio but does not require its investors to devise sophisticated early-exercise strategies. Moreover, the injections to the upgraded fund to replenish the value of the naked fund above the guaranteed level create a leverage on the value of the naked fund which enables the dynamic guaranteed fund investors to

[1] The privatization increases the credit risk of the social security programs, inducing the investors under the privatized programs to seek further financial protection. For example, see the discussion in Pennacchi (1999).

participate in the growth in the fund value in times of good market returns.[2]

Although a dynamic guaranteed fund may not be replicated by a composite structure of basic put and call options[3], it internalizes the call option characteristics by allowing its investor to participate in an upside market with a floor protection and the put option characteristics by leveraging the downside risk with respect to the guaranteed level to enhance the investor ultimate payoff. Gerber and Pafumi (2000) derive a closed-form pricing formula for the continuous dynamic guaranteed fund and the rebalancing portfolio strategy between a risky upgraded fund and a riskless asset for the fund issuers. However, monitoring of the fund is usually at discrete-time instants. Imai and Boyle (2001) investigate the value of the discretely monitored dynamic guaranteed fund using a Monte Carlo simulation approach. In this paper, we develop an analytic pricing formula for the discrete dynamic guaranteed fund, building upon the insight that the total incremental injections depend on the minimum value over the life of the contract, as in the case of pricing a discrete lookback option. Our analytic pricing formula is derived from the risk-neutral expectation approach. We adopt the probability density function of lookback options in AitSahlia and Lai (1998) to calculate the expected payoff, given the possible occurrence of the minimum value over the various discrete time points of the life of the contract. The analytic pricing formula also enables an analytic calculation of the hedging parameters of a synthetic portfolio rebalancing strategy to hedge against the investment risk of the fund issuers. We demonstrate that the delta of our discrete dynamic guaranteed fund valuation model has a call feature, whereas its gamma has a put feature.

Intuitively, a guaranteed fund price process can be viewed as a composition of diffusion and jump process. The diffusion is derived from the standard stochastic dynamic assumption of the underlying naked fund price process, whereas the jumps are consequences of value injections to replenish the fund value above the guaranteed level when needed. We assume the naked fund price process to follow a risk-neutral Brownian motion, and therefore the expected rate of return is the risk-free rate of interest. A hedging error of a delta-hedged portfolio is the costs of establishing the risky and risk-free assets for the current hedging portfolio allocation subtracted by the income from liquidating the risky and risk-free assets brought forward by the previous period portfolio allocation. Without jumps, the delta hedging portfolio allocation between the naked fund and the riskless asset over discrete-time instants will on average lead to zero hedging

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[2] An example is the dynamic guaranteed fund managed by SG Asset Management.

[3] See Gerber and Shiu (2003). This problem leads us to consider dynamically, rather than statically, hedging the dynamic guaranteed fund.

errors.

Since money is injected to replenish the fund to its guaranteed level during discrete dynamic hedging, the injected amount can be translated in dollar terms to upgrade the naked fund price. This upward movement of the fund price causes a sudden rise in the naked fund price to the guaranteed level, i.e., a naked fund price jump. In turn, the jump in relative position alters the stochastic distribution of the terminal value of the guaranteed fund and this adds a differential rate to the risk-free rate of interest. Of course, this rate difference will be wiped out when the guaranteed fund is hedged continuously over its life because the value injection will become infinitesimally small. In the context of dynamic guaranteed fund, the guarantee jump arises from the additional number of naked fund units injected. The random and monotonically increasing number of units of naked fund holding induces the process of hedging errors over its life to have a non-zero expected value. When delta dynamic hedging occurs at discrete instants, the requirement of guarantee protection will be translated in monetary terms into a negative expected total hedging error.

We explore the delta hedging performance by rebalancing hedging portfolio at monthly reset dates over a simulated year using our analytic discrete dynamic guaranteed fund hedging formulation as opposed to the formulation in Gerber and Pafumi (2000). The Monte Carlo simulation results suggest that, using our formula, the average total hedging error is higher and its variance are lower than those based on Gerber and Pafumi (2000). This shows that incompatible model application to discretely price and hedge the dynamic guaranteed fund costs the delta hedger. However, in both cases, the total hedging errors are often negative. Since the hedging amount of the naked fund holding determines the stochastic distribution of the terminal payoff of the hedging portfolio, we can change the stochastic payoff structure by portfolio re-allocation or changing the hedge ratio. As a consequence, the jump sizes, and hence the expected rate of return out of hedging the guaranteed fund over discrete instants are changed. In particular, when the hedge ratio on the risk-free asset is adjusted by a gamma factor, we show that the monetary outlay is substantially reduced. This gamma adjustment amounts to holding additional riskless reserve, against the stochastic guarantee jumps, over that of unadjusted delta hedging throughout the hedging process.

This paper is organized as follows. The next section reviews the theoretical structure of a dynamic guaranteed fund with reference to lookback options in a continuous sampling geometric Brownian motion framework. Section 3 discusses the pricing and hedging formulations of the discrete guaranteed fund in a geometric Brownian motion framework. They are derived by an extension of the pricing formulas of the discrete hindsight options in AitSahlia and Lai (1998). An efficient computational procedure for frequent and/or long maturity monitoring, modifying the Tse, Li and Ng (2001) procedure, is offered for our pricing and hedging of the discrete dynamic guaranteed fund. A synthetic portfolio rebalancing investment strategy in a discrete sampling setting is developed to hedge the investment risk of the fund monitored at discrete time instants. Simulation results are listed in Section 4. It shows the possible divergence of the prices between the discrete and continuous guaranteed fund when monitorings are infrequent. The section also investigates the performance of our synthetic hedging portfolio compared with that of Gerber and Pafumi (2000), and the costs and benefits of hedging performance improvement after modifying the conventional delta by a gamma factor. Section 5 concludes.

2. The Guaranteed Investment Fund and Lookback Options

2.1 The Guaranteed Investment Fund

Gerber and Pafumi (2000) investigate an investment fund with dynamic guarantees. The guarantees are in the form of repeated fund unit injections over the life of the fund to maintain the fund amount at the agreed upon guaranteed level. A fund unit is a unit of account to determine the value of the investor fund holding, i.e., the value of the investor total holding is the price of the fund multiplied by the number of fund units that he holds where the fund price is the per unit value. The naked fund without guarantees is modeled to follow a geometric Brownian motion. Let *F(t)* denote the price of a naked fund at time *t*. Then, its price is represented by

$$F(t) = F(0)e^{\mu t+\sigma W(t)}, t \geq 0 \tag{1}$$

where *F(0)* denotes the naked fund price at time 0, μ is the constant drift rate of the geometric Brownian motion, σ is the constant volatility of the geometric Brownian motion, and *W(t)* is a standard Weiner process. The risk-neutral process under constant risk-free interest rate *r* is

$$d \ln F(t) = \mu * dt + d\widetilde{W}(t), t \geq 0 \tag{2}$$

where $\mu^* = r - \frac{1}{2}\sigma^2$, and $\widetilde{W}(t)$ is a standard Wiener process under the risk-neutral measure. To investigate the dynamic property of the guaranteed fund, Gerber and Pafumi (2000) replace the risk-neutral geometric Brownian process given in equation (2) by an upgraded fund price $\widetilde{F}(t)$ which will receive an injection of the additional number of fund units to replenish the fund price to the constant level *K* whenever the upgraded fund price $\widetilde{F}(t)$ falls below *K*. The upgraded fund price $\widetilde{F}(t)$ is calculated as follows:

$$\widetilde{F}(t) = F(t)n(t) \tag{3}$$

and

$$n(t) = \max\{1, \max_{t \geq s \geq 0} \frac{K}{F(s)}\}$$

where K denotes the constant guaranteed level (or barrier) of the guaranteed investment fund, and $n(t)$ denotes the total number of fund units at time t per one unit of the fund invested at time 0.

Let *A(F(0), T)* denote the discounted expected value of the guaranteed investment fund given in equation (3) at time 0, where T is the maturity date of the guaranteed investment fund. A risk-neutral expectation of the discounted price of the guaranteed fund at maturity time T leads to equation (2.2) in Gerber and Pafumi (2000) which is

$$A(F(0), T) = e^{-rT} E^{*}[\widetilde{F}(T)] \tag{4}$$

where $E^{*}[.]$ represents the expectation operator under the risk-neutral measure given in equation (2). The value of the fund protection at time t ($T \geq t \geq 0$) is

$$V(t) = A(\widetilde{F}(t), T) - \widetilde{F}(t) \tag{5}$$

with $V(T) = 0$.[4] Gerber and Pafumi (2000) derive a closed-form formula as a result of equation (4) as follows:

$$\begin{aligned} A(F(0), T) = {} & Ke^{-rT}\left(1 - \frac{\sigma^2}{2r}\right)\Phi\left(\frac{-\ln(F(0)/K) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ & + \frac{K\sigma^2}{2r}\left(\frac{K}{F(0)}\right)^{\frac{2r}{\sigma^2}}\Phi\left(\frac{-\ln(F(0)/K) + rT + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}\right) \\ & + F(0)\Phi\left(\frac{\ln(F(0)/K) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{6}$$

where $\Phi(.)$ represents the cumulative standard normal distribution function.[5]

2.2 The Lookback Options

Equation (4) suggests that the discounted value of the expected upgraded fund depends on the terminal price of the naked fund *F(T)*. However, the recurrent injections to maintain the price

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[4]The value of the equity forward at time t is $F(t) = e^{-r(T-t)} E^{*}[F(T)]$. Hence, its protection value at any time t prior to the expiration date T is zero, i.e., $V(t) = 0$.

[5] For a continuous dynamic guaranteed fund whose guaranteed level grows exponentially at a constant rate γ i.e., $n(t) = \max\{1, \max_{t \geq s \geq 0} \frac{Ke^{\gamma s}}{F(s)}\}$. Gerber and Pafumi (2000) replace the variables $\widetilde{F}(t)$ by $e^{\gamma t}\widetilde{F}_i(t)$, and $F(t)$ by $e^{\gamma t}F_i(t)$. Thus, $A(F(0), T) = E^{*}\{e^{-rT}\widetilde{F}(T)\} = E^{*}\{e^{-(r-\gamma)T}\widetilde{F}_i(T)\}$. Consequently, the formula for a continuous dynamic guaranteed fund with a guaranteed level growing exponentially at a constant rate γ is the same as equation (6) but the interest rate r is replaced by r-γ.

of the upgraded fund $\widetilde{F}(t)$ above the constant level K at time t, $T \geq t \geq 0$, suggests that it also depends on the history of the evolution path $F(t)$, $T \geq t \geq 0$. Since the sum of the incremental injections to replenish the fund over the finite time period $[0,T]$ is the maximum potential loss over the period, its price must reflect the maximum potential loss of the fund issuer relative to the guaranteed level K over the period $[0,T]$ leading to the lookback option formulation. The formulation

$$\max\{1, \max_{T \geq t \geq 0} \frac{K}{F(t)}\} \tag{7}$$

itself in equation (3) resembles the terminal payoff of a fixed strike lookback call option with respect to the maximum of $K/F(t)$, $T \geq t \geq 0$, and the fixed strike at the value 1. However, the maximum ratio $\max_{T \geq t \geq 0} \frac{K}{F(t)}$ occurs when the random price $F(t)$ attains a minimum over the period $[0, T]$. So, the guaranteed fund has the lookback call option formulation but has the lookback put option characteristic, which protects the investment fund against its downside risk. Because of the downside protection over the life of the dynamic guaranteed fund, we would collect a protection value today commensurate with at least the current early exercise value of the fund. Hence, the fund will never be exercised early to capture a downside protection during a down drift of the naked fund value. Section 3 in Imai and Boyle (2001) provides a proof of this intuition. Hence, rather than discussing American-style dynamic guaranteed funds, this paper only focuses on the pricing and hedging of the European-style dynamic guaranteed fund.

3. Dynamic Guarantees under Discrete Sampling

Whereas continuously sampled closed-form formulas offer exact and efficient valuation of the lookback options, Broadie, Glasserman and Kou (1999), and Heynen and Kat (1995) illustrate the possible mispricing if the closed-form formulas are mistakenly used to price an option which actually is monitored at fixed discrete dates. We develop below analytic pricing and hedging formulas of the discrete dynamic guaranteed fund based on the hindsight option pricing analysis in AitSahlia and Lai (1998). In addition, we introduce an efficient computational procedure which calculates the derived formulas even with frequent monitoring.

3.1 The Discrete Guaranteed Fund Pricing Formula

Let the guaranteed fund be monitored at discrete time points $n\Delta$ for $n = 0,1,\cdots,m,$ where the consecutive time points are separated by equal time interval $\Delta = \frac{T}{m}$, and m denotes the total number of monitorings. The random price of the naked fund at time $n\Delta$ is $F(0)e^{U_n}$, U_n is a random value under stationary Gaussian process at time $n\Delta$ for $n = 0, 1, \cdots, m$ and $U_n \sim \phi(\mu_n, \sigma_n^2)$ where $\mu_n = \mu * n\Delta, \sigma_n = \sigma\sqrt{n\Delta}$, and $\phi(.)$ is a univariate normal probability density function.[6] We denote the minimum value of U_n over the time interval $[0,T]$ by M_m, i.e.,

$$M_m = \min\{U_\nu : m \geq \nu \geq 0\}.$$

The first escape time is defined as the first time (or the minimum amount of time) for a dynamic path to take on a defined characteristic. Let the first escape time for U_ν to become a negative value be τ_- and the first escape time for U_ν to become a positive value be τ_+. Formally,

$$\tau_- = \inf\{\nu : 0 \geq U_\nu\},$$

and

$$\tau_+ = \inf\{\nu : U_\nu > 0\}$$

where $\inf(.)$ is an infimum mathematical operator.

AitSahlia and Lai (1998, page 229, equation (9)) show that the probability that the minimum value M_m falls in the infinitesimal interval $dx (0 \geq x)$ can be calculated by the following equation

$$P(M_m \in dx \mid M_m \neq U_0, U_m) = P(U_1 \in dx) P(\tau_- > m-1) + \sum_{\nu=2}^{m-1} P(\tau_+ > \nu; U_\nu \in dx) P(\tau_- > m-\nu) \quad (8)$$

where $P(.)$ represents the cumulative probability distribution function.[7]

Consistent with our discussion in Section 2, Imai and Boyle (2001) highlight an alternative representation of the terminal price of the upgraded fund in their equation (9) as follows:

$$\widetilde{F}(T) = F(T)\max(1, \widetilde{K}e^{-M_m})$$

where $\widetilde{K} = \frac{K}{F(0)}$ for $\gamma = 0$. Therefore, the value of the discrete guaranteed fund at time 0, $\widetilde{A}(F(0),T)$, is

$$\widetilde{A}(F(0),T) = e^{-rT} E^*[F(T)\max(1, \widetilde{K}e^{-M_m})].$$

The value of the discrete dynamic guaranteed fund protection at time t is

$$\widetilde{V}(t) = \widetilde{A}(\widetilde{F}(t), T-t) - \widetilde{F}(t)$$

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6 Since $F(0)e^{U_0} = F(0)$, it is clear that $U_0 = 0$.

7 $P(\tau_- > m-\nu) = P(U_1 > 0, \cdots, U_{m-\nu} > 0)$ and $P(\tau_+ > \nu; U_\nu \in dx) = P(0 \geq U_1, \cdots, 0 \geq U_{\nu-1}, U_\nu \in dx; 0 \geq x)$. The idea behind their derivation of the probability of a particular observation at time point ν to be a minimum value M_m is that for $U_\nu = M_m$,

$$U_0 \geq U_\nu, \cdots, U_{\nu-1} \geq U_\nu, \text{ and } U_{\nu+1} \geq U_\nu, \cdots, U_m \geq U_\nu.$$

Since the increment $U_i - U_j (i > j)$ are time-independent under Brownian motion, this implies

$$0 \geq U_\nu, 0 \geq U_{\nu-1}, \cdots, 0 \geq U_1, \text{ and } U_1 > 0, U_2 > 0, \cdots, U_{m-\nu} > 0.$$

with $\widetilde{V}(T)=0$.

Equation (8) suggests that equation (9) is composed of a series of expected values to take account of the random minimum value that occurs at various discrete time points. The results of AitSahlia and Lai (1998) reviewed in our Section 3.1 help identify the probability density function of the discrete dynamic guaranteed fund, which is utilized for our risk-neutral expected value calculation. In view of the assumption that the price of the naked fund *F(t)* follows a geometric Brownian motion, the expected value $E^*[\widetilde{F}(T)]$ in equation (9) may be recast into a series of multivariate probability values through applying Proposition 1 in Tse, Li and Ng (2001). In sum, by applying the results of AitSahlia and Lai (1998) and Tse, Li and Ng (2001), we can derive the pricing formula for the value of the discrete dynamic guaranteed fund $\widetilde{A}(F(0),T)$ as follows.[8]

Proposition 1

The pricing formula of the value of the discrete dynamic guaranteed fund $\widetilde{A}(F(0),T)$ over [*0,T*] monitored at *m* time points of equal time subintervals in [*0,T*] is expressed as[9]

$$\widetilde{A}(F(0),T) = F(0)\{\max(0,\widetilde{K}-1)\beta_m + \sum_{\nu=1}^{m}\beta_{m-\nu}[e^{-r\nu\Delta}\widetilde{K}B_\nu^2 - B_\nu^1]+1\} \tag{11}$$

where

$$\beta_0 = 1,$$

$$\beta_\nu = \int_{-\infty}^{d_1(1)}\cdots\int_{-\infty}^{d_1(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu$$

$$B_\nu^1 = \int_{-\infty}^{-d_1(1)}\cdots\int_{-\infty}^{-d_1(\nu-1)}\int_{-\infty}^{-d_1^*(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu$$

$$B_\nu^2 = \int_{-\infty}^{-d_2(1)}\cdots\int_{-\infty}^{-d_2(\nu-1)}\int_{-\infty}^{-d_2^*(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu$$

$\mathbf{x}_\nu'=(x_1,\cdots,x_\nu)$, $m=\frac{T}{\Delta}$, $C_\nu=(2\pi)^{-\frac{\nu}{2}}|Z_\nu|^{-\frac{1}{2}}$, $Z_\nu=[\rho_{ij}]_{i,j=1,\nu}$, $\rho_{i,j}=\frac{i}{\sqrt{ij}}$ for $i,j=1,\cdots,\nu$ and $j\ge i$, Z_ν^{-1} is

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[8] The formula for a discretely sampled lookback put options $\widetilde{L}(F(0),T)$ is

$$\widetilde{L}(F(0),T) = e^{-rT}F(0)\{\max(0,\widetilde{K}-1)\alpha_m + \sum_{\nu=1}^{m}\alpha_{m-\nu}\int_{-\infty}^{-d_2(1)}\cdots\int_{-\infty}^{-d_2(\nu-1)}\int_{-\infty}^{-d_2^*(\nu)}(\widetilde{K}-e^{\sigma_\nu x_\nu+\mu_\nu})C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu\}$$

$$= e^{-rT}F(0)\{\max(0,K-1)\alpha_m + \sum_{\nu=1}^{m}\alpha_{m-\nu}[\widetilde{K}B_\nu^2 - e^{r\nu\Delta}B_\nu^1]\}$$

where $\widetilde{K}=\frac{K}{F(0)}$, $\alpha_0=1$, and $\alpha_\nu=\int_{-\infty}^{d_2(1)}\cdots\int_{-\infty}^{d_2(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu$.

[9] Similar to the discussion in Section 4 of Gerber and Pafumi (2000), the formula for the discrete dynamic guaranteed fund with guaranteed level growing exponentially at constant rate γ can be deduced from replacing r in equation (11) by r-γ.

the inverse of Z_ν, $d_2(i) = \frac{\mu_i}{\sigma_i}$, $d_1(i) = d_2(i) + \sigma_i$, for $i = 1, \cdots, \nu$, $d_2^*(\nu) = \frac{-\nu^* + \mu_\nu}{\sigma_\nu}$,

$d_1^*(\nu) = d_2^*(\nu) + \sigma_\nu$, $v^* = \min[0, \ln(\widetilde{K})]$, and $\widetilde{K} = \frac{K}{F(0)}$.

<u>Proof</u>

See Section A.1 of the Appendix.

The term $e^{rT} F(0) \max(0, \widetilde{K} - 1)\beta_m$ represents the expected protection value when *F(0)* is the minimum value over the life [*0,T*] of the fund. The value β_m is the probability that *F(0)* is the minimum value. The expected protection value is derived on the assumption that the terminal value *F(T)* is not less than the minimum value *F(0)*. However, this expected protection value occurs only when the fund needs to pay $\widetilde{K} - 1$ additional units for the fund protection over the life of the fund. Hence, the fund protection leads to the multiplication factor $\max(0, \widetilde{K} - 1)$.

The term $e^{-rT} F(0) \sum_{\nu=1}^{m} \beta_{m-\nu} [e^{-r\nu\Delta} \widetilde{K} B_\nu^2 - B_\nu^1]$ represents the expected protection value when the minimum value occurs at the 1st to the (*m-1*)th monitoring fixed time points. The value $\sum_{\nu=1}^{m-1} \beta_{m-\nu} B_\nu^2$ denotes the sum of the joint probabilities for the occurrences of the minimum value M_m in the time interval $[\frac{T}{m}, \frac{(m-1)T}{m}]$. The terminal payoff $\max(F(T)\widetilde{K}e^{-M_m} - F(T), 0)$ suggests that additional fund units will be injected for fund protection only when $M_m < \ln(\widetilde{K})$. In addition, the minimum value M_m has to be less than the initial value $U_0 = 0$. Consequently, $M_m < \min[0, \ln(\widetilde{K})]$.

The term $F(0)[\widetilde{K}B_m^2 - e^{rT} B_m^1]$ represents the expected protection value when the minimum value occurs at the *m*th monitoring time point. The probability that the minimum value occurs at the terminal date is B_m^2. The terminal payoff under such an occurrence is $\max(F(0)\widetilde{K} - F(T), 0)$. Since $F(0)\widetilde{K}$ is not random, the payoff $\max(F(0)\widetilde{K} - F(T), 0)$ is of a European put option with strike price $F(0)\widetilde{K}$. This suggests that the dynamic guaranteed fund would guarantee to the least at the strike value $F(0)\widetilde{K}$ even if the minimum value $F(0)e^{M_m} = F(T)$ occurs at the terminal date *T*.

In sum, a dynamic guaranteed fund protects its investor against losses at a strike level $F(0)\widetilde{K}$ but it leverages the fund gain through the contractually required injection of additional fund units based on the minimum value attained over the life of the dynamic guaranteed fund from time 0 to time $\frac{(m-1)T}{m}$. Thus, an investor would not be required to exercise early his fund to capture the gain, unlike in the American-style one-time protection guaranteed fund structure.

3.2 The Imai and Boyle Pricing Formula

While Proposition 1 provides an exact analytic formulation for the value of a discrete dynamic guaranteed fund, Imai and Boyle (2001) establish an alternative formula by applying the approach of Broadie, Glasserman and Kou (1999) to the valuation of a discrete dynamic guaranteed fund. An adjustment based on the continuously monitored option formula is made in Broadie, Glasserman and Kou (1999) to derive approximate formulas for the discrete barrier and lookback options. By applying the Broadie, Glasserman and Kou (1999) adjustment procedure to the continuously monitored guaranteed fund pricing formula of Gerber and Pafumi (2000), Imai and Boyle (2001) derive a similar formula for the value of the discrete dynamic guaranteed fund in their equation (8). On the whole, their pricing formula for a discretely monitored dynamic guaranteed fund modifies the guaranteed level of the continuously monitored dynamic guaranteed fund given in equation (6), and adds a correction term to the modified pricing formula. Without the correction term, the calculation is efficient but “barrier too close” (i.e., error systematically increases as the barrier moves close to the current price level) is found to occur in the pricing of the discrete barrier or lookback options. The correction term is calculated by Monte Carlo simulation. It slows down the computation of the discrete dynamic guaranteed fund value given in equation (8) in Imai and Boyle (2001) and leads to a confidence interval to capture the actual value. Our analytic formula based on the risk-neutral expectation given in equation (11) is an exact valuation formulation. However, its degree of accuracy still hinges on an appropriate numerical procedure for delivering accurate probability values. We discuss below an efficient numerical procedure that can calculate probability values with high precision. The relative performance of the two formulas is discussed in Section 4. Although the results of Imai and Boyle (2001) are not as fine as our results, they provide a means of validating the authenticity of our formula given in equation (11) for pricing a discrete dynamic guaranteed fund.

3.3 A Modified Numerical Procedure

We modify the numerical procedure in Tse, Li and Ng (2001) for our discrete dynamic guaranteed fund calculation, by imposing a fixed integral cutoff limit and an unscaled multivariate probability distribution on their original procedure. To value the discrete dynamic guaranteed fund, an efficient computational procedure is necessary for two reasons. First, our discrete dynamic guaranteed fund formula given in equation (11) requires the consideration of the occurrence of minimum naked fund price at every monitoring date, which could lead to a long series of expected value calculations. Second, a fund by its nature is usually of longer maturity than an option. This characteristic can proportionally increase the number of monitorings.

Therefore, an efficient computational technique is necessary to cut down on computational time while retaining an acceptable degree of accuracy. Tse, Li and Ng (2001) find that the inverse of the full rank square coefficient matrix of the random state variables generated under Brownian motion is a tridiagonal matrix. Based on this matrix, they derive the numerical efficiency from exploiting the recursive functional structure in the valuation of their discrete barrier and hindsight options. They demonstrate that the order of computational complexity of a discretely sampled financial derivatives under geometric Brownian motion is $(p+(m-1)p^2)$ where p is the number of computational points per integral of the Gaussian quadrature method. However, p grows nonlinearly in value when the number of monitorings m increases. In terms of computational complexity, our modified procedure is similar to that of Tse, Li, and Ng (2001). However, the number of computational points per layer, p, grows roughly linearly. So, this enables the calculation of fairly accurate probability and/or expected payoff value of frequent monitoring in a much shorter time.

3.3.1 Error Analysis of the Modified Numerical Procedure

The modified Tse, Li and Ng (2001) procedure proposed by us trades error control for efficiency. We identify the numerical integrity of the modified procedure by the following proposition.

Proposition 2

For a continuous function

$$\mathrm{G}(\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{m}) = \mathrm{G}_1(\mathrm{x}_1)\,\mathrm{G}_2(\mathrm{x}_1, \mathrm{x}_2) \cdots \mathrm{G}_\mathrm{m}(\mathrm{x}_{\mathrm{m-1}}, \mathrm{x}_\mathrm{m}),$$

the absolute error between its m-variate integral and its m-variate numerical sum is bounded below by $m\,\varepsilon$, i.e.,

$$\mathrm{m}\,\varepsilon \geq \left| \int_{I_1} \cdots \int_{I_m} G(x_1, \cdots, x_m)\,\mathrm{d}\mathbf{x}_\mathrm{m} - \sum_{\mathrm{x}_\mathrm{m}} \mathrm{G}^*_\mathrm{m}(\mathbf{x}_\mathrm{m}) \right| \tag{12}$$

where

$$\varepsilon = \left| \int_{\mathrm{I}_\mathrm{i}} \mathrm{G}(\mathrm{x}_{\mathrm{i-1}}, \cdots, \mathrm{x}_\mathrm{i})\,\mathrm{dx}_\mathrm{i} - \sum_{\mathrm{x}_\mathrm{i}} \mathrm{G}^*_\mathrm{i}(x_{i-1}, x_i) \right|,$$

and is a constant, |.| is an absolute value operator, I_i is the integral domain for the random variable x_i, G_i is a recursive functional of the random variables x_{i-1} and x_i and G is a continuous functional of the random variables $x_1, \cdots, x_m$.

Proof

See Section A.2 of the Appendix.

Proposition 2 is useful for error estimate only if $\varepsilon = |\int_{I_i} G(x_{i-1}, \cdots, x_i)\, dx_i - \sum_{x_i} G_i^*(x_{i-1}, x_i)|$. This implies that the integration limit and the numerical integration error for the underlying function over the integration range per layer should be the same. Under this fixed limit, Proposition 2 suggests that the overall numerical errors including cutoff and numerical integration errors will at most grow at a linear rate as *m* increases. With this proposition, an efficient calculation of a fairly accurate discrete hedging formula discussed below using our modified Tse, Li and Ng (2001) procedure is supported.

3.4 The Synthetic Hedging Strategy

To protect *A*(*F*(*0*), *T*) at level *K* for a duration [*0,T*], Gerber and Pafumi (2000) introduce the idea of replicating portfolio as a hedging strategy to replicate the terminal value of the dynamic guaranteed fund which requires the issuer to continuously rebalance his portfolio between the risky asset $\widetilde{F}(t)$ and a riskless asset.[10] The strategy calls for an allocation at time *t* the amount

$$\widetilde{F}(t) A_f(\widetilde{F}(t), T-t)$$

to the risky asset, and the amount

$$A(\widetilde{F}(t), T-t) - \widetilde{F}(t) A_f(\widetilde{F}(t), T-t)$$

to the riskless asset, where $A_f(.)$ denotes the partial derivative (or delta δ_t) of $A(\widetilde{F}(t), T-t)$ with respect to $\widetilde{F}(t)$.[11] As in the continuous sampling formulation of Gerber and Pafumi (2000), we deduce our synthetic portfolio rebalancing strategy under discrete sampling which allocates at time *t* an amount

$$\widetilde{F}(t) A_f(\widetilde{F}(t), T-t)$$

to the risky asset $\widetilde{F}(t)$, and an amount

$$A(\widetilde{F}(t), T-t) - \widetilde{F}(t) \widetilde{A}_f(\widetilde{F}(t), T-t)$$

to the riskless asset. Without loss of generality, we set *T-t* = *T'*. Therefore, $\widetilde{A}_f(\widetilde{F}(t), T-t)$ is now

[10] Unlike the portfolio rebalancing hedging strategy of the Black-Scholes European options which corrects the option values with respect to the initial hedging portfolio by the changes in the naked investment fund and riskless asset holdings throughout the fund life, the rebalancing portfolio is re-constructed with respect to the upgraded investment fund and riskless asset each hedging period in Gerber and Pafumi (2000). Since *F(t)* instead of $\widetilde{F}(t)$ is tradeable, the actual portfolio holding of the risky asset at any time *t* is $n(t) A_f(\widetilde{F}(t), T-t)$ units of *F(t)*.

[11] For synthetic investment fund protection of the continuously sampled dynamic guaranteed fund, see equations (5.6) to (5.7) in Gerber and Pafumi (2000).

equal to $\widetilde{A}_f(\widetilde{F}(t),T')$. The value $\widetilde{A}_f(\widetilde{F}(t),T')$ can be derived as follows.[12]

$$\widetilde{A}_f(\widetilde{F}(t),T') = 1 - \sum_{\nu=1}^{m'} \beta_{m'-\nu} B_\nu^1 \qquad \text{for } 1 \geq \widetilde{K} \tag{13}$$

where

$$\beta_0 = 1,$$
$$\beta_\nu = \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu{}' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu,$$
$$B_\nu^1 = \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(\nu-1)} \int_{-\infty}^{-d_1^*(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu{}' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu,$$
$$B_\nu^2 = \int_{-\infty}^{-d_2(1)} \cdots \int_{-\infty}^{-d_2(\nu-1)} \int_{-\infty}^{-d_2^*(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu{}' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu,$$

$\mathbf{x}_\nu{}' = (x_1,\cdots,x_\nu), m' = \frac{T'}{\Delta}, C_\nu = (2\pi)^{-\frac{\nu}{2}} |Z_\nu|^{-\frac{1}{2}}, Z_\nu = [\rho_{ij}]_{i,j=1,\nu}, \rho_{i,j} = \frac{i}{\sqrt{ij}}$ for $i,j = 1,\cdots,\nu$ and $j \geq i, Z_\nu^{-1}$ is the inverse of $Z_\nu, d_2(i) = \frac{\mu_i}{\sigma_i}, d_1(i) = d_2(i) + \sigma_i$, for $i = 1,\cdots,\nu$, $d_2^*(\nu) = \frac{-\nu^* + \mu_\nu}{\sigma_\nu}, d_1^*(\nu) = d_2^*(\nu) + \sigma_\nu$, $\nu^* = \ln(\widetilde{K})$, and $\widetilde{K} = \frac{K}{F(t)}$.

Based on the above analytic delta formula, we can also calculate the analytic gamma (Γ_t) formula which is the second partial derivatives of $\widetilde{A}(\widetilde{F}(t),T')$ with respect to $\widetilde{F}(t)$. The gamma $\widetilde{A}_{f^2}(.)$ formula is

$$\widetilde{A}_{f^2}(\widetilde{F}(t),T') = -\sum_{\nu=1}^{m'} \beta_{m'-\nu} b_\nu^1 \qquad \text{for} \quad 1 \geq \widetilde{K} \tag{14}$$

where

$$\beta_0 = 1,$$
$$\beta_\nu = \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(\nu)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu{}' Z_\nu^{-1}\mathbf{x}_\nu} d\mathbf{x}_\nu,$$
$$b_\nu^1 = \frac{1}{\sigma_\nu \widetilde{F}(t)} \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(\nu-1)} C_\nu e^{-\frac{1}{2}\mathbf{x}_\nu^*{}' Z_\nu^{-1}\mathbf{x}_\nu^*} d\mathbf{x}_{\nu-1},$$

---

[12] Merton (1976) recognizes that a perfect hedge for the jump risk may not be practically feasible. Since the risks are not completely hedged, he discusses (in his page 132) how the portfolio returns under this hedging strategy are distributed between the periods with and without jumps. Whereas the expected rate of loss due to jumps in Merton (1976) is offset by the mean rate of returns of the diffusion process, our upgraded fund price process does not have the same constraint. In fact, the frequency and the size of the jumps in our guaranteed price structure depend on how often and how much a geometric Brownian motion of the naked fund prices fall below a pre-set guaranteed level. These structural differences create an opportunity for hedging improvement.

$\mathbf{x}_\nu^{*}{}' = (x_1, \cdots, x_{\nu-1}, -d_1^{*}(\nu)).$

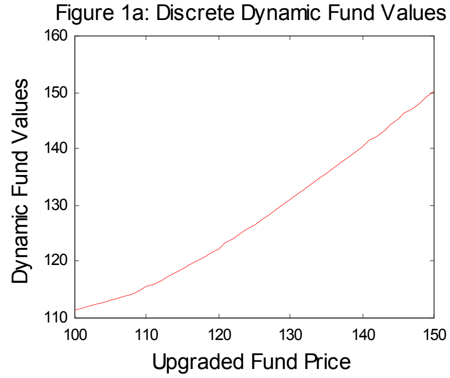


Figure 1a: Discrete Dynamic Fund Values

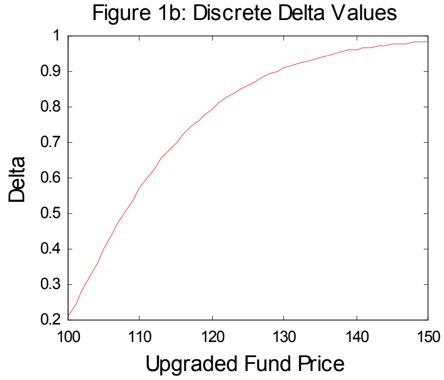


Figure 1b: Discrete Delta Values

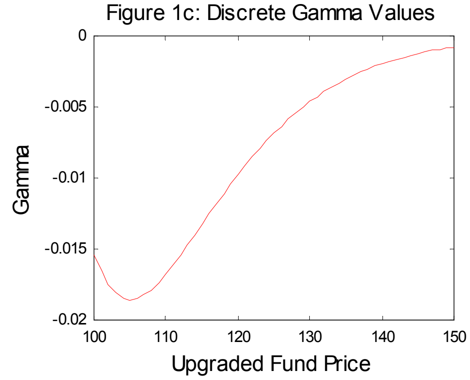


Figure 1c: Discrete Gamma Values

In Figure 1a, we plot the values of the discrete dynamic guaranteed fund with the guaranteed level K set at *100*. They increase with the increase in the upgraded fund price $\tilde{F}(t)$. Figure 1b shows that the delta value is between 0 and 1. Hence, it is a delta of a call feature. In Figure 1c, the gamma value is negative suggesting that it is a gamma of a put feature. From the discussion in Section 3.1, the value of downside protection also has a put feature. Consequently, we can infer that the protection value of the discrete dynamic guaranteed fund may be captured by the curvature of our discrete dynamic guaranteed fund valuation model with respect to the upgraded fund price $\tilde{F}(t)$. Since the price of an equity forward under risk neutral expectation is $F(t) = e^{-r(T-t)} E^{*}[F(T) \mid F(t)]$, we can infer that the protection value is the discounted expected value of the price upgrading or the price jumps to remain above the guaranteed level, i.e., $\tilde{V}(t) = e^{-r(T-t)}\{E^{*}[\tilde{F}(T)|F(t)] - E^{*}[F(T)|F(t)]\}$. In the next section, we demonstrate that a simple delta dynamic hedging creates negative total hedging error. Thus, adjusting the delta by a gamma factor in our model for two-asset delta hedging may retrieve the protection value caused by the jump injections of the discrete dynamic guaranteed fund which was lost during the unadjusted delta dynamic hedging of the fund. Figure 1c shows that gamma is most negative when the upgraded fund price is around the guaranteed level. This is consistent with our intuition that the jump impact on our valuation model is the largest around the guaranteed level and then diminishes when the upgraded fund price moves further above the guaranteed level. Thus, the gamma formula is useful for our coming discussion of retrieving the protection value of the discrete dynamic guaranteed fund.

3.5 A Gamma Adjustment of Delta-Hedged Portfolio

In this section, we investigate the impact of gamma adjustment of the delta on the discrete dynamic hedging performance of a guaranteed fund. As above, we require a delta hedging portfolio consisting of risky and riskless assets to replicate the terminal value of the fund. However, the portfolio allocation strategy between the risky and riskless assets need not be a self-financing strategy. Under discrete dynamic hedging, this period hedging error comes from the difference between the income brought forward from liquidating the preceding period hedging portfolio and the cost of constructing the current period hedging portfolio. The total hedging error is the sum of individual period hedging errors. Next, we introduce the gamma adjustment factor.

Since the protection value may have lost in the delta dynamic hedging process, adjusting the delta by a gamma factor, which captures the value of downside protection, may to a large extent reduce the expected total hedging error. To simplify notations, we denote $\widetilde{F}(t)$, $\widetilde{A}(\widetilde{F}(t), T\text{-}t)$, $\widetilde{A}_f(\widetilde{F}(t), T\text{-}t)$, $n(i\varDelta)$ respectively by $\widetilde{F}_t$, $\widetilde{c}_t$, δ_t, and $n_{i\Delta}$. The ith period hedging error after a gamma adjustment of the delta hedging portfolio at the (i-1)th period is

$$HI_{i\varDelta}(\theta_g) \;=\; \delta_{(i\text{-}1)\varDelta}\widetilde{F}_{i\varDelta} + \; e^{r\varDelta}\,[\widetilde{c}_{(i\text{-}1)\varDelta}\text{-}(\delta_{(i\text{-}1)\varDelta} + \theta_g \varGamma_{(i\text{-}1)\varDelta}\sigma^2 \widetilde{F}_{(i\text{-}1)\varDelta})\widetilde{F}_{(i\text{-}1)\varDelta}]\text{ - }\widetilde{c}_{i\varDelta}. \tag{15}$$

where $\theta_g = 0$ and $\frac{1}{2}$ respectively for an unadjusted and a gamma-adjusted delta hedging portfolio.[13]

As a whole, the expected value of the total hedging error can be summarized by a linear equation

$$E^*[\sum_{i=1}^{m} HI_{i\varDelta}(\theta_g)] \;=\; E^*[\sum_{i=1}^{m} HI_{i\varDelta}(0)]\text{ -}\theta_g e^{r\varDelta}\sigma^2\; E^*[\sum_{i=1}^{m} \varGamma_{(i\text{-}1)\varDelta}\widetilde{F}^2_{(i\text{-}1)\varDelta}]. \tag{16}$$

---

[13] Unlike the plain vanilla Black-Scholes European options, the underlying asset of a guaranteed fund is usually not a tradeable asset. In particular, the upgraded fund is the underlying asset of the dynamic guaranteed fund. In reality, the two-asset delta dynamic hedging portfolio can only be constructed by its closest tradeable surrogate risky asset: the naked investment fund. The value $\widetilde{F}_{i\Delta}$ in the first term given in equation (15) then has to be replaced by $n_{(i-1)\Delta}F_{i\Delta}$ to reflect the hedging difficulty caused by the non-tradeability of the upgraded fund.

Equation (15) is consistent with the Ito's lemma in that it adjusts the discrete dynamic guaranteed fund value by delta and gamma with respect to the upgraded fund $\widetilde{F}(t)$. However, the dynamic hedging is implemented under a two-asset portfolio of risky and riskless assets. In the next section, numerical experiment suggests that $E^*[\sum_{i=1}^{m} HI_{i\Delta}(0)] < 0$ reflecting that the issuer has to pay a protected rate of return of the guaranteed fund higher than the risk-free rate of interest. Since the gamma value $\Gamma_{(i-1)\Delta}$ is negative, $\frac{1}{2}\Gamma_{(i-1)\Delta}\sigma^2\widetilde{F}_{(i-1)\Delta}$ in equation (15) cuts down the delta value $\delta_{(i-1)\Delta}$ in the riskless asset amounting to holding additional riskless reserve to capture the rate differential over the risk-free rate of interest induced by the guarantees. However, there does not have a gamma adjustment at the maturity date T because its protection value is zero. At this date, the delta value is equal to *1*. Hence, the maturity value of the discrete dynamic guaranteed fund is replicated by the value of the risky asset of the hedging portfolio. Altogether, the gamma adjustment of the delta leads to an increase in value of the expected total hedging error given in equation (16).

So far, our discussion of hedging improvement relies on the general structure of guaranteed funds, and therefore, it applies liberally to a wide variety of guaranteed funds rather than to the dynamic guaranteed fund alone.[14] Under discrete dynamic hedging and by using Monte Carlo simulation, we will study examples for the liquidity requirement, the individual period hedging errors, and the distribution of the total hedging error before and after gamma adjusted delta hedging asset re-allocation in the next section.

4. Simulations

Simulation studies are implemented to investigate the accuracy of our formula in calculating the value of dynamic guaranteed fund protection compared with that in Imai and Boyle (2001). We then explore the hedging performance using our analytic discrete dynamic guaranteed fund hedging formulation instead of the continuous dynamic guaranteed fund hedging formula in Section 5 of Gerber and Pafumi (2000), when monitorings over the simulated year occur at monthly time points. Finally, we investigate some properties arising after a gamma adjustment of a delta-hedging portfolio.

To facilitate the investigation of the hedging performance of a discrete dynamic guaranteed fund and its tradeoffs, we require a reasonably fast numerical procedure to deliver accurate pricing and

[14] Equity-linked notes have similar valuation structure. For example, consider the principal-guaranteed equity-linked note $\widetilde{F}_t = \max(F_t, K)$ which can be expressed as $\widetilde{F}_t = F_t n(t)$ where $n(t) = \max(1, \frac{K}{F_t})$.

hedging results of the guaranteed fund. The Tse, Li and Ng (2001) procedure with fixed cutoff limit provides a reasonable environment for Proposition 2 to apply because it not only has a fixed cutoff limit but also allows us to control the numerical integration errors by the computational point estimation formula given in equation (12) in Tse, Li and Ng (2001). To identify the efficiency and the degree of accuracy of our results, we calculate individual probability values benchmarked by the counterpart values using the limit-adjusting procedure given in Tse, Li and Ng (2001).

Table 1

A Comparison of Probability Values[a]

m[b]	-1.5[c]		0		1.5	
	TLN[d]	TLN[e]	TLN	TLN_f	TLN	TLN_f
1	*066807201219*	*0.066807201261*	*0.499999999950*	*0.499999999993*	*0.933192798681*	*0.933192798724*
	(0.032)[f]	*(0.000)*	*(0.032)*	*(0.000)*	*(0.031)*	*(0.000)*
2	*0.028590881592*	*0.028590881577*	*0.375000000000*	*0.374999999985*	*0.894976479054*	*0.894976479040*
	(0.032)	*(0.000)*	*(0.031)*	*(0.000)*	*(0.016)*	*(0.000)*
7	*0.005738568673*	*0.005738568628*	*0.209472656250*	*0.209472656202*	*0.809452586866*	*0.809452586817*
	(0.083)	*(0.063)*	*(0.078)*	*(0.063)*	*(0.093)*	*(0.063)*
12	*0.002836380302*	*0.002836380241*	*0.161180257797*	*0.161180257724*	*0.770331628955*	*0.770331628881*
	(0.344)	*(0.126)*	*(0.343)*	*(0.160)*	*(0.343)*	*(0.155)*
36	*0.000666743355*	*0.000666743274*	*0.093705675297*	*0.093705675153*	*0.692132063952*	*0.692132063798*
	(11.162)	*(1.478)*	*(11.145)*	*(1.163)*	*(11.121)*	*(1.289)*
52	*0.000409848596*	*0.000409848515*	*0.078051172374*	*0.078051172203*	*0.667031883073*	*0.667031882885*
	(46.475)	*(2.514)*	*(46.475)*	*(2.577)*	*(59.505)*	*(2.646)*

[a]The probability values are calculated assuming that the underlying stochastic processes follow Brownian motion.

[b]The values *m* stand for the number of monitorings.

[c]We use the same upper limit values for all the monitoring dates.

[d]The values under columns TLN are computed based on the computational procedure in Tse, Ng and Li (2001). We set the cutoff error tolerance at 10^{-10} and the numerical integration error tolerance at 10^{-13} for our calculation using this procedure.

[e]The probability values calculated under column TLN_f are based on the computational procedure in Tse, Ng and Li (2001) with fixed cutoff limit. We set the cutoff at *-6.75* and the numerical integration error tolerance at 10^{-13} for our calculation using this procedure.

[f]The values in parentheses underneath their probability values are the computational time in seconds for computing these probability values using MATLAB version 6.5 software and 1.6GH Pentium IV personal computer.

In Table 1, the numerical figures under columns *TLN* are computed by the variable cutoff-limit numerical procedure in Tse, Li and Ng (2001), whereas the numerical figures under the columns TLN_f are computed by the fixed cutoff-limit modified procedure of Tse, Li and Ng

(2001). The computational errors of the probability values under columns *TLN* are controlled to be accurate to 10 decimal places. By choosing the lower cutoff limit at -6.75, the Tse, Li and Ng (2001) probability values under fixed limit are accurate to 9 decimal places with 52 monitorings. A comparison of the corresponding figures under columns *TLN* and TLN_f shows that the computational errors grow roughly at a linear rate in accordance with the growth in the number of monitorings *m*. Thus, the probability values with 260 monitorings should be accurate to at least 7 decimal places. Therefore, the 4-decimal place numerical figures reported in Tables 2 to 4 are exact provided that equations (11) and (13) provide the correct pricing and hedging formulas of the discrete dynamic guaranteed fund.

4.1 Analytic Valuation of Dynamic Guaranteed Fund Protection

To justify the validity of our valuation formula given in equation (11) for exact pricing of discrete dynamic guaranteed funds, we compare the numerical results of the protection values derived from our valuation formula under similar terms with those in Imai and Boyle (2001). In Table 2, the values under column CDGFP are continuously sampled dynamic guaranteed fund protection values calculated using the analytic formula in Gerber and Pafumi (2000). The values under column MC are Monte Carlo simulation results copied from Table 10 of Imai and Boyle (2001). The approximate formula of Imai and Boyle (2001) is derived from an extension of Broadie, Glasserman and Kou (1999). Since the correction term of their discrete dynamic guaranteed fund pricing formula is computed by Monte Carlo simulation, a confidence interval is reported. The values of the discrete dynamic guaranteed fund protection under column DDGFP are calculated using our analytic formula given in equation (11).

First, our values are consistently smaller when the number of monitorings declines from weekly monitoring to monthly monitoring. This conforms to our intuition that the amount of expected total guarantee injections decreases when the number of monitorings is reduced. Compared with the numerical intervals of Imai and Boyle (2001), our numerical values for the discrete dynamic guaranteed fund values stay within their intervals for short maturity and close-to-money. Those of our values that go out of their numerical ranges are higher but at most by a second decimal place difference. Whereas their continuous formula correction approach is conceptually different from our discrete probability valuation approach, the consistency of the two sets of results mutually support both conceptual frameworks. However, with our analytic valuation formula, it enables an easy derivation of the analytic delta- and gamma- hedging formulas of the discrete dynamic guaranteed fund. In the following, we explore the hedging performance using the continuous formulas in Gerber and Pafumi (2000) and our discrete analytic valuation and hedging formulas under the modified numerical procedure of Tse, Li and Ng (2001).

Table 2

Monte Carlo versus Analytic Valuation[a]

T	K	CDGFP[b]	Weekly		Monthly	
			MC[c]	DDDFP[d]	MC	DDGFP
1	*100*	14.7931	13.053	13.0389	11.375	11.3608
			12.977		11.096	
	90	6.0120	5.196	5.1801	4.461	4.4446
			5.121		4.197	
	80	1.7709	1.494	1.4811	1.254	1.2414
			1.451		1.119	
3	*100*	23.8741	21.993	21.9430	20.060	20.0089
			21.915		19.890	
	90	13.4646	12.338	12.2866	11.194	11.1429
			12.253		10.999	
	80	6.6443	6.054	6.0054	5.357	5.3966
			5.981		5.295	
5	*100*	29.1716	27.130	27.1462	25.097	25.0915
			27.097		25.021	
	90	18.0257	16.709	16.7063	15.395	15.3963
			16.682		15.294	
	80	10.1373	9.340	9.3441	8.559	8.5645
			9.326		8.487	

[a]Option parameters: $F(0) = 100$, $r = 0.04$, $\gamma = 0$, and $\sigma=0.2$.

[b]CDGFP denotes the continuously sampled dynamic guaranteed fund protection values. These are calculated using the formula in Gerber and Pafumi (2000).

[c]MC denotes the Monte Carlo results copied from Imai and Boyle (2001), Table 10.

[d]DDGFP denotes the discretely sampled dynamic guaranteed fund protection values. These values are calculated using the formula in equation (11).

Figure 2a: Naked and Upgraded Fund Prices and Hedging Amount in Risky Asset

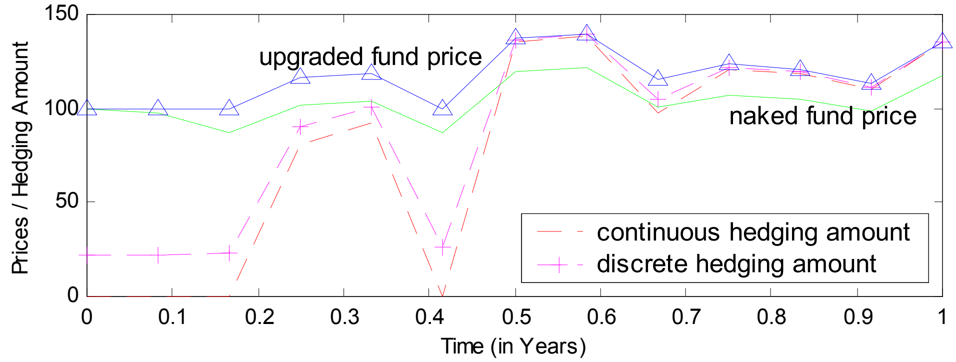
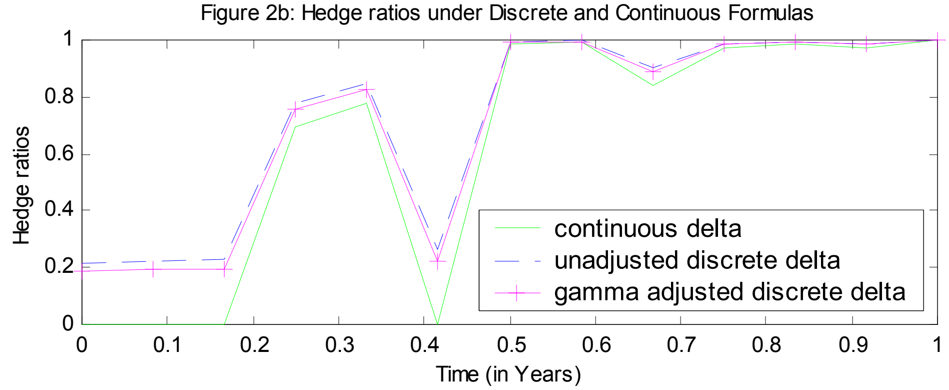
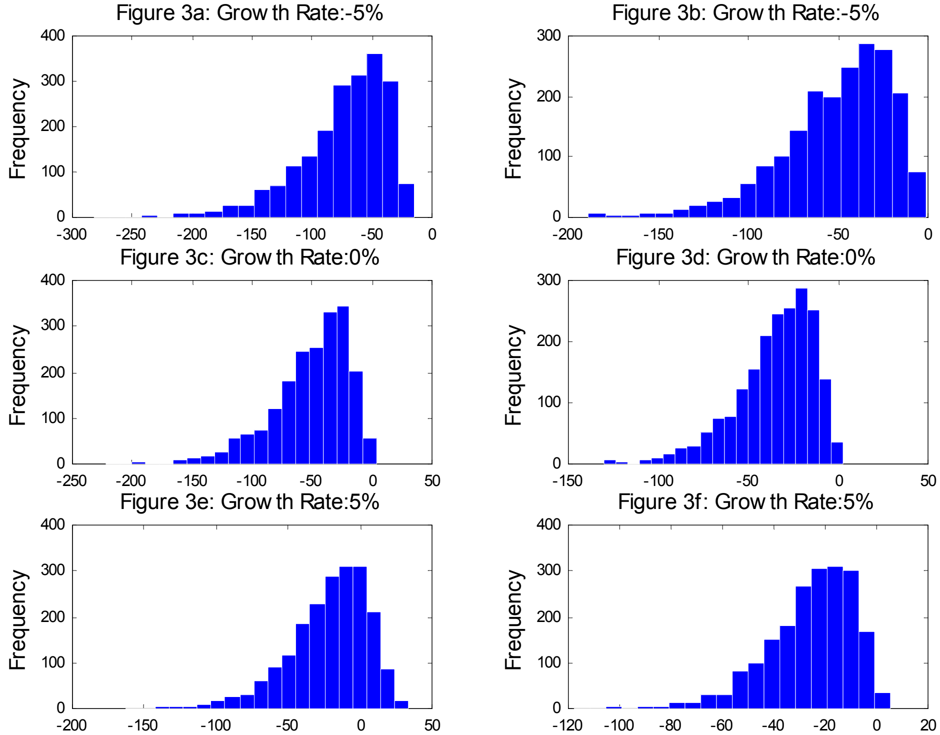


Figure 2b: Hedge ratios under Discrete and Continuous Formulas



Figures 3a,3c,3e: Continuous Distribution

Figures 3b,3d,3f: Discrete Distribution



4.2 The Hedging Errors of Dynamic Guaranteed Funds

Based on Section 5 in Gerber and Pafumi (2000), we compute the amount invested in the risky reference portfolio $\boldsymbol{R_C}$ and the amount invested in the risk free asset RF_C at the fixed time point t. We calculate the continuous dynamic guaranteed fund hedging error HI_C, which is the costs of risky and risk free assets this period subtracted by the income of previous period risky and risk free assets brought forward.[15] In contrast, our formulation in equation (13) suggests an alternative synthetic portfolio under discrete sampling consisting of the risky reference portfolio $R_{\boldsymbol{D}}$ and a risk-free reserve $RF_{\boldsymbol{D}}$ to hedge against the uncertainty of the upgraded fund $\widetilde{F}(t)$. Over time, however, the patterns of fluctuations of the delta hedge ratios under the discrete and continuous hedging formulas are similar, i.e., the values rise and fall together (see Figure 2b). Under the continuous hedging formula in Gerber and Pafumi (2000), the delta hedge ratio will be zero whenever the upgraded fund price is at the guaranteed level.[16] This is not the case under our discrete hedging formula whose hedge ratio is always above the ratio calculated under the continuous hedging formula in Gerber and Pafumi (2000). This zero hedge ratio effect creates wide gyration in the hedging amount of the risky asset (see Figure 2a) which can suddenly move between zero to values close to the upgraded fund prices. Nonetheless, the total hedging error using our discrete hedging formulation $\Sigma_t\ HI_{\boldsymbol{D}}$ has less deviation from zero than using the continuous hedging formula of Gerber and Pafumi (2000) when both formulas are applied to calculate the rebalancing portfolios which are rebalanced only at the monthly reset dates.

Figure 3 plots the frequency distributions of the total hedging errors $\Sigma_t\ HI_C$ and $\Sigma_t\ HI_D$ for $\gamma = -0.05$, 0, and 0.05. The frequency distributions are generated by 2000 Monte Carlo simulation runs of $F(t)$ over the simulated year monitored at monthly instants under the same parameters as in Section 4.2. Each run creates one data point for $\Sigma_t\ HI_D$ and $\Sigma_t\ HI_C$. Figures 3a, 3c, and 3e are for $\Sigma_t\ HI_C$ with $\gamma = -0.05$, 0, and 0.05 respectively, whereas Figures 3b, 3d, and 3f are for $\Sigma_t\ HI_D$ with $\gamma = -0.05$, 0, and 0.05 respectively. The means of the distributions in Figures 3a, 3c, 3e are: -74.2875, -50.6406, and -21.4703. Their standard deviations are: 38.5418, 32.4409, and 27.5801 respectively. The means of the distributions in Figures 3b, 3d, 3f are: -50.6092, -34.8035, and -25.7475. Their standard deviations are: 30.4608, 21.6993, and 17.0885 respectively.

In all these cases, the variances of the frequency distributions of $\Sigma_t\ HI_{\boldsymbol{C}}$ and their absolute

[15] We assume zero hedging error in the initial period for delta dynamic hedging. Usually a synthetic hedging portfolio is created when the corresponding option is sold. Consequently, the initial option value covers the cost of synthetic portfolio creation.

[16] This usually occurs when the naked fund price is drifting down. If this continues, the guaranteed fund pays the non-stochastic guaranteed fund price K at maturity. This is the rationale for holding risk-free asset only before fund maturity when the upgraded fund price is at the guaranteed level K.

values of the means are in general higher than those of $\Sigma_t\ HI_{\boldsymbol{D}}$. These indicate a reduced absolute expected total hedging error and its variability for discrete dynamic hedging when our analytic hedging formula instead of the Gerber and Pafumi (2001) formula is applied to calculate the rebalancing portfolio at the discrete hedging time points. Hence, the model misspecification of using continuous rather than discrete valuation and hedging formulas for discrete dynamic hedging is translated into a monetary outlay. Consequently, we investigate how a gamma adjustment of the hedge ratio on the riskless asset can cut down the absolute expected hedging errors under our discrete hedging formulation in equation (13). Such investigation is important because Figure 3 suggests that the guaranteed fund issuer to delta-hedge its fund will end up losing money most of the time regardless of using the continuous or discrete hedging formulas. Without an alternative hedging mechanism to adjust the expected total hedging error, establishing and running a dynamic guaranteed fund is an economically irrational and infeasible business even for a risk-neutral fund issuer.

Table 3

Hedging Performance of the Synthetic Investment Fund

Under Delta Hedging, and Gamma-adjusted Delta Hedging[a]

t	F(t)[b]	n(t)[c]	$\widetilde{F}(t)$ [d]	Delta Hedging: $\theta_g = 0$[e]			Adjusted Delta Hedging: $\theta_g = \frac{1}{2}$		
				RF_D[f]	R_D[g]	HI_D[h]	RF_D	R_D	HI_D
	100.0000	*1.0000*	*100.0000*	*89.5188*	*21.8420*	*0*	*92.6411*	*18.7197*	*3.1223*
1/12	*97.6962*	*1.0236*	*100.0000*	*88.3381*	*22.4790*	*0.8426*	*91.6107*	*19.2065*	*3.9753*
2/12	*87.5762*	*1.1419*	*100.0000*	*87.0345*	*23.2097*	*0.8679*	*90.4768*	*19.7674*	*4.1514*
3/12	*101.7688*	*1.1419*	*116.2060*	*27.7990*	*90.6322*	*-4.1351*	*30.8719*	*87.5593*	*-0.6813*
4/12	*104.0691*	*1.1419*	*118.8326*	*19.7436*	*100.4986*	*0.3304*	*22.1611*	*98.0811*	*3.4135*
5/12	*86.9640*	*1.1499*	*100.0000*	*82.0326*	*26.2688*	*-3.9203*	*86.1448*	*22.1566*	*-1.4947*
6/12	*119.5328*	*1.1499*	*137.4509*	*1.2505*	*136.2560*	*-19.0933*	*1.4109*	*136.0956*	*-14.9674*
7/12	*121.3264*	*1.1499*	*139.5134*	*0.4692*	*139.0618*	*0.0242*	*0.5190*	*139.0120*	*0.1851*
8/12	*100.7214*	*1.1499*	*115.8196*	*11.9745*	*104.4259*	*-0.4848*	*13.5456*	*102.8548*	*-0.4348*
9/12	*107.4321*	*1.1499*	*123.5363*	*1.6739*	*121.9184*	*-0.1945*	*1.8105*	*121.7819*	*1.3819*
10/12	*104.9749*	*1.1499*	*120.7108*	*0.9639*	*119.7724*	*0.0731*	*0.9872*	*119.7491*	*0.2101*
11/12	*98.2732*	*1.1499*	*113.0045*	*1.5867*	*111.4499*	*0.0566*	*1.5867*	*111.4499*	*0.0800*
1	*117.9580*	*1.1499*	*135.6400*	*0*	*135.6400*	*-0.2740*	*0*	*135.6400*	*-0.2740*
Hedging Error:				$\Sigma_t HI_D =$ *-25.9072*			$\Sigma_t HI_D =$ *-1.3326*		

[a]Option parameters: $r = 0.04$, $\sigma = 0.2$, $\gamma = 0$, $K = 100$, $\Delta = 1/12$, and $T = 1$.

[b]*F(t)* is the naked fund value at time *t* which follows a geometric Brownian motion, i.e., $F(t) = F(0)e^{U_t}$, $U_t \sim \phi(\mu^* t, \sigma^2 t)$. This specific stochastic series is generated by Monte Carlo stochastic simulation.

[c] $n(t) = \max\{1, \max_{0 \geq s \geq t} \frac{Ke^{\gamma s}}{F(s)}\}$.

[d] $\widetilde{F}(t) = n(t)F(t)$.

[e]Columns 5 to 7 are for $\theta_g = 0$, whereas columns *8* to *10* are for $\theta_g = 1/2$.

[f]$RF_D(t)$ denotes the amount invested in the riskless asset.

[g]$R_D(t)$ denotes the amount invested in the risky asset *F(t)*.

[h] $HI_D(t) = [e^{r\Delta} RF_D(t-\Delta) - RF_D(t)] + [\frac{R_D(t-\Delta)F(t)}{F(t-\Delta)} - R_D(t)]$

4.2.1 The Gamma-Adjusted Delta Hedging

The numerical figures in Table 3 are primarily calculated based on formulas in this article and the established formulas in the literature of finance. They are calculated under the same conditions and parameters as in Section 4.2. Moreover, all the pricing and hedging numerical figures in Table 3 are derived based on one and the same arbitrarily chosen sample path of this risk-neutral naked fund price process. To understand the impact of the gamma hedging parameter on the hedging errors, we set θ_g=1/2 for gamma-adjusted delta hedging. Our purpose of gamma adjustment of the delta hedge ratio is to retrieve the protection value, which was not captured by the delta portfolio allocation during our dynamic hedging.[17]

In Table 3, we compare the hedging performance between the delta ($\theta_g = 0$) hedging and the gamma-adjusted delta (θ_g =1/2) hedging of the dynamic guaranteed fund at monthly time instants over a simulated year. The dynamic path of the naked fund is generated by Monte Carlo simulation under the chosen set of parameters of interest rate, volatility, time-to-maturity, and the guaranteed level in Section 4.2. We calculate the number of units $n(t)$ and the upgraded fund price $\widetilde{F}(t)$ at time t by equation (3) where $T \geq t \geq 0$. According to Gerber and Pafumi (2000), a synthetic fund investment can be split into an investment in the reference portfolio R_D and an investment in the risk free asset RF_D. Our simulation run in Table 3 suggests that the total hedging error of this synthetic hedging portfolio $\Sigma_t HI_D$ for $\gamma = 0$ under delta hedging is -25.9072 and under gamma-adjusted delta hedging is -1.3326. Thus, the gamma adjustment retrieves some of the protection value due to the guarantee jumps. After this adjustment, the effective delta for the riskless asset is lower because of the put characteristics of the gamma of the dynamic guaranteed fund. Figure 2b shows that the adjustment is more substantial when the upgraded fund price is at the guaranteed level at which the jump injection has the highest impact. Nonetheless, the effective delta still falls between 0 and 1 so that the amount of risk-free asset remains to be positive throughout the simulated year. Although the pattern of fluctuations between the delta and the effective delta hedge ratios are similar, the adjustment causes sometimes opposite movement of the hedging errors HI_D between the delta and gamma-adjusted delta hedging in Table 3 at various time points of the simulated-year, and hence a difference in their total hedging errors.

To investigate the distribution of the hedging errors, we simulate 2000 stochastic paths of the naked fund price under the same terms as in Table 3. Figures 4a, 4c, and 4e show the distributions of the total hedging errors which correspond to those in Figures 4b, 4d, and 4f. Figures 4b, 4d, and 4f compare on the same graph the root mean squared errors (*RMSE*) of the individual month hedging errors over the simulated year between the delta hedging and the gamma-adjusted delta hedging. The means in Figures 4a, 4c, 4e are -13.9315, 0.1758, and 5.5016, and their standard deviations are 31.5924, 22.0377, and 18.7941 respectively. Figures 4b, 4d, and

[17] Note that we are not utilizing gamma hedging in the usual sense for minimizing the variability of hedging errors created by the curvature of the original pricing model, which will require a third asset with option feature to hedge the nonlinearity.

4f suggest that maximum *RMSE* of delta and of gamma-adjusted delta hedging are more or less the same.[18] Consequently, the gamma-adjusted delta hedging retrieves the protection value without incurring additional costs of substantial borrowing, and additional variability of total and individual period hedging errors.[19]

5. Conclusions

This paper investigates the analytic valuation and hedging strategy of discrete dynamic guaranteed fund. The pricing and hedging formulations are based on the expected payoff approach. Competition among insurance companies leads to many innovative product designs to offer their investors higher returns and protect them against the downside risk of these returns. In this respect, a dynamic guaranteed fund is ideal, and does not require active monitoring by the investor of the fund value to make prudent early-exercise and rollover decisions. While continuously sampled option formula offers an efficient reference for the value of an option, a discretely sampled option formula is the theoretically accurate formulation for practical applications.

A continuously sampled dynamic guaranteed fund formula is derived by Gerber and Pafumi (2000), and Gerber and Shiu (2003). Because a discrete dynamic guaranteed fund is practically more important, Imai and Boyle (2001) provide an approximate formula by extending the formulation of Broadie, Glasserman and Kou (1999). Since a discretely sampled option value could lead to a large variation from its continuous counterpart (see Kat and Verdonk (1995)), an analytic pricing formulation of the discrete dynamic guaranteed fund can provide an exact value of the fund and an understanding of its hedging performance. As the terminal value of a dynamic guaranteed fund is leveraged by its injections of incremental units to upgrade the naked fund value to remain above the guaranteed level over its life, the total value injection depends on the minimum value of the naked fund during its life. Therefore, we make use of the probability density function for discrete lookback options in AitSahlia and Lai (1998) to develop our pricing and hedging formulas for the discrete dynamic guaranteed fund. Our formulas are a result of the expected terminal value of the upgraded fund conditional on the possible occurrence of the

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[18] The maximum *RMSE* in Figures 4b, 4d, and 4f are 44.5006, 28.4116, 27.5696 respectively for delta hedging, and 40.4143, 29.1603, 20.8104 respectively for gamma-adjusted delta hedging.

[19] When the naked fund surrogate is used to capture non-tradeability of the fund in place of the upgraded fund for our discrete dynamic hedging, the means corresponding to Figures 4a, 4c, 4e are -23.5906, -12.8082, and -9.8931 respectively. Since our hedging model is constructed based on the underlying upgraded fund price, the use of the tradeable naked fund surrogate for discrete dynamic hedging implies a model misspecification which manifests itself as a hedging error.

minimum value at various fixed discrete time instants.

Imai and Boyle (2001) provide a pricing interval for the discrete dynamic guaranteed fund. However, we are able to accurately calculate its analytic value contingent on the degree of accuracy of the required probability value input. Furthermore, based on our Monte Carlo simulation experiments, the expected total hedging error for a delta synthetic portfolio rebalancing strategy over a year for hedging the dynamic guaranteed fund at monthly monitoring time points is lower in value and higher in variance using the continuously sampled hedging formula of Gerber and Pafumi (2000) than using our discretely sampled hedging formulation. Nonetheless, the requirement of guarantee injections leads to negative expected total hedging error under delta dynamic hedging of the discrete dynamic guaranteed fund. By applying a gamma-adjusted hedge ratio to the riskless asset, a fund issuer can to a large extent retrieve the protection value of the guaranteed fund during its dynamic hedging. In addition, our paper shows that this gamma-adjusted delta hedging strategy based on a two-asset hedging portfolio of the naked fund and the riskless asset does not incur additional costs of substantial borrowing, and additional variability of hedging errors over the unadjusted delta hedging strategy.

In sum, this paper furthers our understanding of the value of dynamic guaranteed fund protection compared with other related forms of protection. It demonstrates how we can improve the hedging performance of a guaranteed fund through modifying the conventional delta by a gamma factor.

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Appendix

A.1 Proof of Proposition 1.

Let us define

$$d_2^*(v) = \frac{-v^* + \mu_v}{\sigma_v},$$

$$d_1^*(v) = d_2^*(v) + \sigma_v,$$

$$d_2(v) = \frac{\mu_v}{\sigma_v},$$

$$d_1(v) = d_2(v) + \sigma_v,$$

for $\nu = 1, \cdots, m$ where $v^* = \min[0, \ln(\widetilde{K})]$. Also, we define

$$\beta_v = \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(v)} C_v \, e^{-\frac{1}{2}\mathbf{y}_v' Z_v^{-1} \mathbf{y}_v} d\mathbf{y}_v$$

and where $C_v = (2\pi)^{-v/2} |Z_v|^{-1/2}$, Z_v is a $\nu \times \nu$ full rank matrix arising from Brownian motion, $\mathbf{y}_v' = (y_1, \cdots, y_v)$.

Under risk neutral expectation, the value of the discrete guaranteed fund at time 0 is

$$\begin{aligned}
\widetilde{A}(F(0),T) &= e^{-rT} E^*[F(T)\max(\widetilde{K}\, e^{-M_m}, 1)] \\
&= e^{-rT} E^*[F(T)\max(\widetilde{K}\, e^{-M_m} - 1, 0) + F(T)] \\
&= e^{-rT} F(0)\{E^*[e^{U_m} E^*(\max(\widetilde{K}\, e^{-x} - 1, 0)|x = M_m)|U_m = M_m] \\
&\quad + E^*[e^{U_m} E^*(\max(\widetilde{K}\, e^{-x} - 1, 0)|x = M_m)|U_m > M_m] + E^*[e^{U_m}]\} \\
&= e^{-rT} F(0)\{\int_{U_m}^{\infty} \cdots \int_{U_m}^{\infty} \int_{-\infty}^{0} e^{U_m} (\max\ (\widetilde{K}\, e^{-U_m} - 1, 0)) f(U_m|U_m = M_m)\, g(U_m = M_m)\, d\mathbf{U} \\
&\quad + \sum_{i=1}^{m-1} \int_{U_i}^{\infty} \cdots \int_{U_i}^{\infty} \int_{-\infty}^{0} \int_{U_i}^{\infty} \cdots \int_{U_i}^{\infty} e^{U_m} (\max\ (\widetilde{K}\, e^{-U_i} - 1, 0)) f(U_m|U_i = M_m)\, g(U_i = M_m)\, d\mathbf{U} \\
&\quad + \int_0^{\infty} \cdots \int_0^{\infty} e^{U_m} (\max\ (\widetilde{K}\, e^{-U_0} - 1, 0)) f(U_m|U_0 = M_m)\, g(U_0 = M_m)\, d\mathbf{U} + E^*[e^{U_m}]\}.
\end{aligned}$$

where $f(U_m \mid U_i = M_m)$ is the probability density function of U_m conditional on the occurrence of minimum M_m at U_i, and $g(U_i = M_m)$ is the normal probability density function of U_i under Brownian motion. Since $U_0 = 0$ and $e^{-rT} E^*[e^{U_m}] = 1$, by applying the joint probability results in AitSahlia and Lai (1998), the value of the discrete guaranteed fund can therefore be written as

$$\widetilde{A}(F(0),) = e^{-rT} F(0)\{\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\int_{-\infty}^{\min(0,\ln\widetilde{K})}(\widetilde{K} - e^{U_m})g(0 > U_{m-i}, U_m \in dU_m; i < m)d\mathbf{U}_{i,m}$$

$$+\sum_{i=1}^{m-1}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\int_{-\infty}^{\min(0,\ln\widetilde{K})} e^{U_i}(\widetilde{K}e^{-U_i} - 1)g(0 > U_{i-j}, U_i \in dU_i; j < i)d\mathbf{U}_{1,i} \times$$

$$\int_{0}^{\infty}\cdots\int_{0}^{\infty} e^{U_{m-i}} g(U_{k-i} > 0; k > i)d\mathbf{U}_{1,m-i}$$

$$+\max(\widetilde{K} - 1, 0)\int_{0}^{\infty}\cdots\int_{0}^{\infty} e^{U_m} g(U_k > 0; k > 0)d\mathbf{U}_{1,m} + e^{rT}\}$$

$$= e^{-rT} F(0)\{\int_{-\infty}^{-\frac{\mu_1}{\sigma_1}}\cdots\int_{-\infty}^{-\frac{\mu_{m-1}}{\sigma_{m-1}}}\int_{-\infty}^{\frac{v^*-\mu_m}{\sigma_m}}(\widetilde{K} - e^{\sigma_m y_m + \mu_m})C_m e^{-\frac{1}{2}\mathbf{y}_m' Z_m^{-1}\mathbf{y}_m} d\mathbf{y}_m$$

$$+\sum_{i=1}^{m-1}\int_{-\infty}^{-\frac{\mu_1}{\sigma_1}}\cdots\int_{-\infty}^{-\frac{\mu_{i-1}}{\sigma_{i-1}}}\int_{-\infty}^{\frac{v^*-\mu_m}{\sigma_m}}(\widetilde{K} - e^{\sigma_i y_i + \mu_i})C_i e^{-\frac{1}{2}\mathbf{y}_i' Z_i^{-1}\mathbf{y}_i} d\mathbf{y}_i \times$$

$$\int_{-\frac{\mu_1}{\sigma_1}}^{\infty}\cdots\int_{-\frac{\mu_{m-i}}{\sigma_{m-i}}}^{\infty} e^{\sigma_{m-i} y_{m-i} + \mu_{m-i}} C_{m-i} e^{-\frac{1}{2}\mathbf{y}_{m-i}' Z_{m-i}^{-1}\mathbf{y}_{m-i}} d\mathbf{y}_{m-i}$$

$$+\max(\widetilde{K} - 1, 0)\int_{-\frac{\mu_1}{\sigma_1}}^{\infty}\cdots\int_{-\frac{\mu_m}{\sigma_m}}^{\infty} e^{\sigma_m y_m + \mu_m} C_m e^{-\frac{1}{2}\mathbf{y}_m' Z_m^{-1}\mathbf{y}_m} d\mathbf{y}_m + e^{rT}\}$$

$$= F(0)\{1 + \max(0, \widetilde{K} - 1)\beta_m$$

$$+\sum_{v=1}^{m} e^{-rv\Delta}\beta_{m-v}\int_{-\infty}^{-d_2(1)}\cdots\int_{-\infty}^{-d_2(v-1)}\int_{-\infty}^{-d_2^*(v)}(\widetilde{K} - e^{\sigma_v y_v + \mu_v})C_v e^{-\frac{1}{2}\mathbf{y}_v' Z_v^{-1}\mathbf{y}_v} d\mathbf{y}_v\}$$

By applying the results of Lemma 1, we obtain

$$\widetilde{A}(F(0), T) = F(0)\{1 + \max(0, \widetilde{K}-1)\beta_m$$

$$+\sum_{v=1}^{m} e^{-rv\Delta}\widetilde{K}\beta_{m-v}\int_{-\infty}^{-d_2(1)}\cdots\int_{-\infty}^{-d_2(v-1)}\int_{-\infty}^{-d_2^*(v)} C_v e^{-\frac{1}{2}\mathbf{y}_v' Z_v^{-1}\mathbf{y}_v} d\mathbf{y}_v$$

$$-\sum_{v=1}^{m}\beta_{m-v}\int_{-\infty}^{-d_1(1)}\cdots\int_{-\infty}^{-d_1(v-1)}\int_{-\infty}^{-d_1^*(v)} C_v e^{-\frac{1}{2}\mathbf{y}_v' Z_v^{-1}\mathbf{y}_v} d\mathbf{y}_v\}$$

Q.E.D.

A.2 Proof of Proposition 2.

The proof will be furnished upon request.