Loglinear approximate solutions to RBC models:

An illustration and some observations

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Abstract

Following the analytical approach suggested in Campbell (1994), this paper considers a baseline real-business-cycle (RBC) model with endogenous labor supply. It is observed that the coefficients in the loglinear equations approximating the equilibrium are related to the fundamental parameters in a relatively simple manner. These equations can be utilized to obtain the closed-form approximate solution with ease and to demonstrate the properties (say, uniqueness) of the solution with clarity. Furthermore, comparative static results can be confirmed analytically (by, for example, straightforward differentiation). We believe that (at least some of) these conclusions can be generalized to more complicated RBC models.

Keywords: Real-business-cycle models; Loglinear approximate solution

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1. INTRODUCTION

The real-business-cycle (RBC) theory has emerged as an important approach in macroeconomics in the past two decades. In a typical macroeconomic paper using the RBC approach (such as King and Rebelo, 1999), agents with well-articulated preferences and opportunities optimize in a dynamic and stochastic environment. The system of dynamic expectational equations characterizing the equilibrium consistent with the agents' optimizing behavior is then analyzed with respect to the specific economic issues being studied.

While the above idea is conceptually simple, a practical problem of the RBC models is that, in general, there is no closed-form solution, due to the interaction of linear and nonlinear elements (see, for example, Campbell, 1994; Romer, 2001). One commonly used solution procedure is computational, and this poses no major problem since computational methods are no longer costly. The advantage of the computational approach is that it can easily be applied to different models, including those with a large number of state variables and/or stochastic shocks (see, for example, Uhlig, 1999).

In a widely cited paper, Campbell (1994, p. 464) argues that the computational approach, while possessing the advantage mentioned above, is often "mysterious to the noninitiate." He proposes an analytical solution procedure for RBC models based on loglinear approximation,[1] and argues that this approach makes it easier for the reader to understand the mechanism of a particular model being analyzed.[2]

The analytical and computational approaches can be regarded as complementary,

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[1]Specifically, Campbell (1994) suggests taking a first-order Taylor approximation in logs of the variables for the relevant equations. He also explains why, in a homoskedastic setting, this method gives the same results as Christiano's (1988) method of taking a second-order Taylor approximation.

[2]This approach has also been used in other papers, including Campbell and Ludvigson (2001), Lau (2002), and Lettau (2003).

and each has its own merits. The computational method is applicable to many RBC models but the intuition may not be clearly seen in some cases. On the other hand, if the problem can be solved by an analytical method, then the intuition is usually more easily seen. However, the disadvantage is that the analytical approach works satisfactorily only for small and easy-to-manage models. In this context, this paper has a simple objective—making some observations about the analytical approach by using a baseline RBC model as an illustration. For a relatively easy-to-manage dynamic model in which the analytical approach is used, our results suggest that (a) the coefficients in the loglinear approximate equations are related to the underlying parameters in a relatively simple manner (after using the relationships along the balanced growth path to simplify); (b) by focusing on a small set of state variables, it is possible to obtain an approximate solution in closed form and show some desirable properties (such as uniqueness) of the solution; and (c) comparative static results (such as monotonic relationships with respect to the underlying parameters) can be demonstrated analytically. These are desirable properties that warrant the effort required to obtain the loglinear approximate solution.

This paper is organized as follows. In Section 2, we describe the setup of a baseline RBC model and obtain the first-order conditions. In Section 3, we loglinearize the system of expectational difference equations characterizing the equilibrium and derive the solution in closed form. In Section 4, we demonstrate that in the deterministic version of the baseline RBC model, the (approximate) rate of convergence of output per worker towards the balanced growth path is monotonically increasing with respect to the exponent of labor in the production function. We provide concluding remarks in Section 5.

2. A BASELINE RBC MODEL AND THE FIRST-ORDER CONDITIONS

We examine a baseline RBC model, which has been considered in most graduate-level macroeconomics textbooks (see, for example, Romer, 2001). It is a discrete-time one-sector neoclassical model of capital accumulation augmented by endogenous labor supply and stochastic productivity shocks, and it has been studied by many researchers, including Prescott (1986), King et al. (1988), Campbell (1994), and King and Rebelo (1999).

The closed economy we consider is populated by a large number of infintely-lived agents (household-producers), and the population size is assumed to be constant (and normalized to 1) for simplicity. The representative agent is endowed with one unit of time at each period, which is split into work (N) and leisure ($1 - N$). The agent's objective is to maximize expected lifetime utility

$$E_t \sum_{j=0}^{\infty} \beta^j U \left(C_{t+j}, 1 - N_{t+j} \right), \tag{1}$$

where β $(0 < \beta < 1)$ is the discount factor, C_{t+j} is consumption at period $t + j$, and E_t represents the expectation operator conditional on the period-t information set.

Following Campbell (1994) and King and Rebelo (1999), the momentary utility function is given by

$$U \left(C_t, 1 - N_t \right) = \ln C_t + \theta \frac{(1 - N_t)^{1-\gamma} - 1}{1 - \gamma}, \tag{2}$$

where γ (≥ 0) is the reciprocal of the elasticity of intertemporal substitution for leisure, and θ (> 0) is a leisure preference parameter, which is chosen to match the steady-state value of leisure. The above specification is motivated by the results in King et al. (1988) that when the momentary utility function is additively separable in consumption and leisure, log utility for consumption is required to obtain constant

labor supply on the balanced growth path. On the other hand, while the form of the utility function for leisure is not restricted by the balanced growth requirement, power utility is chosen as it nests two popular cases in the RBC literature: log utility for leisure in a model with divisible labor, and linear derived utility for leisure in a model with indivisible labor suggested by Hansen (1985) and Rogerson (1988). The former case corresponds to the limiting case that γ tends to 1, and the latter corresponds to $\gamma = 0$.

The agents in this economy produce by combining labor and the service of capital (K). The production technology is represented by a constant-returns-to-scale Cobb-Douglas production function[3]

$$Y_t = A_t[(1+g)^t N_t]^\alpha K_t^{1-\alpha} = A_t(1+g)^{\alpha t} N_t^\alpha K_t^{1-\alpha}, \tag{3}$$

where $0 < \alpha < 1$, $g > 0$, and Y_t stands for output. In the above specification, the exogenous technology can be split into two parts. The deterministic labor-augmenting component, $(1+g)^t$, grows exogenously at constant rate g. This represents permanent technological variations. On the other hand, the stochastic component A_t leads to temporary change in total factor productivity. Furthermore, we assume that A_t satisfies the following first-order autoregressive process

$$(\ln A_t - \ln A) = \phi(\ln A_{t-1} - \ln A) + \varepsilon_t, \tag{4}$$

where ϕ $(0 \leq \phi \leq 1)$ is an autocorrelation parameter which measures the persistence of the process, and $\ln A$ is the unconditional mean of $\ln A_t$. The random variable ε_t corresponds to a white noise disturbance.

The intertemporal resource constraint is described by:

$$K_t = (1-\delta)K_{t-1} + Y_{t-1} - C_{t-1}, \tag{5}$$

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[3]The production function is neoclassical—exhibiting positive and diminishing marginal products to each input, constant returns to scale, and satisfying Inada conditions.

where δ $(0 \leq \delta \leq 1)$ is the depreciation rate per period, and the initial level of capital is given. The current capital stock is equal to previous undepreciated capital $(1 - \delta)K_{t-1}$ plus previous saving $Y_{t-1} - C_{t-1}$. It is easy to see from (5) that capital stock at period t, K_t, is chosen at time $t - 1$ and is therefore predetermined at time t.

Along the (stochastic) balanced growth path of this economy, output, consumption and capital (but not labor input) all grow at the average rate of g. We can transform this growing economy into a stationary one by dividing the trending variables Y_t, C_t, K_t by the growth component $(1 + g)^t$. This leads to two changes in the above model. First, the transformed lifetime utility function, after ignoring an unimportant constant term, becomes

$$E_t \sum_{j=0}^{\infty} \beta^j \left[\ln c_{t+j} + \theta \frac{(1 - N_{t+j})^{1-\gamma} - 1}{1 - \gamma} \right], \tag{6}$$

where a transformed variable (e.g., $c_t = \frac{C_t}{(1+g)^t}$) is denoted by its lowercase counterpart. Second, the intertemporal resource constraint (5), after combining with (3) and dividing by $(1 + g)^t$, is modified to

$$(1 + g)k_{t+1} = (1 - \delta)k_t + A_t N_t^{\alpha} k_t^{1-\alpha} - c_t. \tag{7}$$

The first-order conditions for optimal choice of this model, given the objective function (6), and the constraints (4) and (7), are given by:

$$\frac{1}{c_t} = \frac{\beta}{(1+g)} E_t \left[\frac{1 - \delta + (1 - \alpha)A_{t+1}N_{t+1}^{\alpha}k_{t+1}^{-\alpha}}{c_{t+1}} \right], \tag{8}$$

and

$$\theta(1 - N_t)^{-\gamma}c_t = \alpha A_t N_t^{-(1-\alpha)} k_t^{1-\alpha}. \tag{9}$$

Equation (8) is an intertemporal efficiency condition. The left-hand side represents the marginal cost in terms of utility of investing one more unit of capital. The right-hand side describes the expected marginal utility gain. At the margin, the cost and

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benefit must be equalized. Equation (9) equates the marginal rate of substitution between consumption and labor to the marginal product of labor. It is an intratemporal efficiency condition which represents the underlying labor-leisure trade-off.

The equilibrium path of this economy, in terms of consumption, capital, labor, and technology shock, is described by the system of non-linear expectational difference equations: (4), (7), (8) and (9). The equilibrium path also satisfies the following transversality condition:

$$\lim_{j \to \infty} \beta^j \left(\frac{1}{c_{t+j}} \right) k_{t+j+1} = 0. \tag{10}$$

3. THE LOGLINEAR APPROXIMATE SOLUTION

It is well known that an exact analytical solution is usually not available in RBC models due to the mixture of loglinear equations like (9) and linear equations like intertemporal resource constraint (7).[4] Here we follow the approach in Campbell (1994) to provide an approximate analytical solution to the above problem. We take a loglinear approximation of the system of expectational equations (4), (7), (8), and (9) around the balanced growth path in terms of the original variables (or equivalently, around the steady-state values in terms of the transformed variables). Provided that the stochastic technology shock does not perturb the economy so that it greatly deviates from its balanced growth path, the loglinear approximation will be a good one.

Setting $A_t = A$, $c_t = c$, $k_t = k$, $N_t = N$ for all t, the following equations—derived respectively from equations (7), (8) and (9)—define the non-stochastic steady-state

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[4]Exact analytical solution is known to exist in RBC models with log utility and complete one-period depreciation, namely $\delta = 1$. For examples, please refer to Long and Plosser (1983), McCallum (1989), and Romer (2001). Complete depreciation within one period, however, is regarded by many as highly unrealistic.

values (without time subscript)

$$(1 + g)k = (1 - \delta)k + AN^\alpha k^{1-\alpha} - c, \tag{7a}$$

$$1 + g = \beta[1 - \delta + (1 - \alpha)AN^\alpha k^{-\alpha}], \tag{8a}$$

and

$$\theta(1 - N)^{-\gamma}c = \alpha AN^{-(1-\alpha)}k^{1-\alpha}. \tag{9a}$$

In the Appendix, we linearize (7), (8) and (9) around the steady-state values ($\ln c$, $\ln k$, $\ln N$, and $\ln A$) and obtain

$$\widehat{N}_t = \lambda_{na}\widehat{A}_t - \lambda_{nc}\widehat{c}_t + \lambda_{nk}\widehat{k}_t, \tag{11}$$

$$\widehat{k}_{t+1} = \lambda_{kk}\widehat{k}_t + \lambda_{ka}\widehat{A}_t + \lambda_{kn}\widehat{N}_t - \lambda_{kc}\widehat{c}_t, \tag{12}$$

$$\widehat{c}_t = E_t\left(\widehat{c}_{t+1} - \lambda_{ca}\widehat{A}_{t+1} - \lambda_{cn}\widehat{N}_{t+1} + \lambda_{ck}\widehat{k}_{t+1}\right), \tag{13}$$

where $\widehat{z}_t = \ln z_t - \ln z$ ($z = A, N, k, c$) represents the percentage deviation of the variable from its steady-state value, and the coefficients are defined as[5]

$$\lambda_{na} = \lambda_{nc} = \frac{1}{1 - \alpha + \gamma\left(\frac{N}{1-N}\right)} > 0, \tag{14}$$

$$\lambda_{nk} = (1 - \alpha)\lambda_{na} > 0, \tag{15}$$

$$\lambda_{kk} = \frac{1}{\beta} > 0, \tag{16}$$

$$\lambda_{ka} = \frac{1 - \beta(1 - \delta_g)}{\beta(1 - \alpha)} > 0, \tag{17}$$

$$\lambda_{kn} = \alpha\lambda_{ka} > 0, \tag{18}$$

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[5]The underlying parameters of the RBC model considered here are β, γ, θ, α, g, δ, and ϕ. Following the RBC literature, some parameters are chosen to match the steady-state values of important variables. We follow Campbell (1994) in choosing β to make the steady-state value of real interest rate equal to 6% per annum and choosing θ to make $N = \frac{1}{3}$. Reflecting this procedure, the "λ" coefficients in (14) to (21) are expressed in terms of N instead of θ.

$$\lambda_{kc} = \frac{1 - \beta(1 - \alpha\delta_g)}{\beta(1 - \alpha)} > 0, \tag{19}$$

$$\lambda_{ca} = [1 - \beta(1 - \delta_g)] > 0, \tag{20}$$

$$\lambda_{cn} = \lambda_{ck} = \alpha\lambda_{ca} > 0, \tag{21}$$

and

$$\delta_g \equiv \frac{\delta + g}{1 + g}. \tag{22}$$

Note that δ_g can be interpreted as the effective depreciation rate of capital in terms of the transformed variables. Because $0 \le \delta \le 1$ and $g > 0$, we know that δ_g in (22) is in the closed interval $[0, 1]$. Note also that (4) is already loglinear and there is no need to approximate it.

So far, we have approximated a system of nonlinear expectational difference equations in consumption, labor, capital stock and technological shock by a system of loglinear expectational difference equations comprising (4), (11), (12) and (13). We next solve this system of equations.

A number of solution methods for a linear (or loglinear) system of expectational equations exist. To facilitate comparison with Campbell (1994) and Lau (2002), we use the method of undetermined coefficients. First, we use (4), (11), (12) and (13) to obtain a second-order expectational difference equation in capital, (A9) in the Appendix. Then, we conjecture the solution of the capital stock in terms of its lag and the technology shock as

$$\widehat{k}_{t+1} = \eta_{kk}\widehat{k}_t + \eta_{ka}\widehat{A}_t, \tag{23}$$

where η_{kk} (partial elasticity of capital with respect to its lag term) and η_{ka} (partial elasticity of capital with respect to lag technology shock) are unknown constants to be determined.

The two unknown coefficients are obtained as follows. Equate coefficients on $\widehat{k}_t$ of

(A10) in the Appendix to find η_{kk}. In the Appendix, we show that

$$\eta_{kk} = \frac{(1 + Q_1 Q_2 + Q_3 Q_4) - \sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2}}{2 Q_1}, \qquad (24)$$

where the coefficients are given by

$$Q_1 = (1 + \lambda_{cn} \lambda_{nc}) = 1 + \alpha \left[\frac{1 - \beta(1 - \delta_g)}{1 - \alpha + \gamma \left(\frac{N}{1-N} \right)} \right] > 1, \qquad (25)$$

$$Q_2 = (\lambda_{kk} + \lambda_{kn} \lambda_{nk}) = \frac{1}{\beta} \left\{ 1 + \alpha \left[\frac{1 - \beta(1 - \delta_g)}{1 - \alpha + \gamma \left(\frac{N}{1-N} \right)} \right] \right\} > \frac{1}{\beta} > 1, \qquad (26)$$

$$Q_3 = (\lambda_{kc} + \lambda_{kn} \lambda_{nc}) > 0, \qquad (27)$$

and

$$Q_4 = (\lambda_{ck} - \lambda_{cn} \lambda_{nk}) = \alpha \left[1 - \beta(1 - \delta_g) \right] \left[\frac{\gamma \left(\frac{N}{1-N} \right)}{1 - \alpha + \gamma \left(\frac{N}{1-N} \right)} \right] > 0. \qquad (28)$$

Once η_{kk} is found, we can equate coefficients for $\widehat{A}_t$ of (A10) to obtain

$$\eta_{ka} = \frac{(\lambda_{ka} + \lambda_{kn} \lambda_{na})(1 - Q_1 \phi) + (\lambda_{ca} + \lambda_{cn} \lambda_{na}) Q_3 \phi}{1 + Q_1 (Q_2 - \eta_{kk} - \phi) + Q_3 Q_4}. \qquad (29)$$

The loglinear approximate solution of the whole system is given by (23) to (29). Given the initial level of capital and a sequence of realized values of technology shock according to (4), we can use (23) to (29) and other equations of the model to simulate the dynamic path of various variables.

4. RATE OF CONVERGENCE

One hotly debated empirical topic in the last decade is how fast output per worker (or output per capita) converges to its balanced growth path.[6] For example, according

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[6]Besides the empirical controversy, the debate about convergence rate also has a long history in the theoretical growth literature (see, for example, the different views in R. Sato, 1963 and K. Sato, 1966) since it is "the crucial determinant of the relevance of the steady state relative to the transitional path." (Turnovsky, 2002, p. 1766.)

to the empirical results in Barro and Sala-i-Martin (1992), the convergence rate across 48 contiguous US states as well as in cross-country data is roughly 2% per year, much slower than the speed predicted by the standard neoclassical growth model. On the other hand, by incorporating human capital, Mankiw et al. (1992) find that countries converge at about the rate predicted by the augmented Solow model.

In this section, we study the rate of convergence of output per worker towards the balanced growth path (or equivalently, output per efficiency unit of labor towards the steady state) in the deterministic version of the baseline RBC model. As in other sections of this paper, our focus is methodological. Specifically, we aim to demonstrate that it is a straightforward (though somewhat tedious) task to show analytically that the (approximate) rate of convergence increases monotonically in the exponent of labor in the production function (α), an important parameter of the model.[7] (The value of $1 - \alpha$, the exponent of capital in the production function, is closely related to the convergence debate.) Since it is more convenient to see the relationship in an environment with no stochastic shock, we examine the deterministic version of the baseline RBC model (with $A_t = 1$, $\forall t$) in the remaining analysis.

To relate the rate of convergence to α, our first step is to express the percentage deviation of output per efficiency unit of labor, $(\widehat{y}_t - \widehat{N}_t)$, as a function of its lagged terms. It is easy to show that the relationship is given by a first-order difference

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[7]Our analysis of the convergence rate is complementary to Turnovsky (2002) in two ways. First, Turnovsky (2002) uses more general utility and production functions, and focuses on the effects of various intertemporal and intratemporal elasticities of substitution on the convergence rate, while we focus on the effect of the exponent of labor in the Cobb-Douglas production function. Second, Turnovsky (2002) uses a numerical approach to obtain the convergence rate, whereas our paper— with its methodological orientation—uses an analytical approach.

equation[8]

$$\left(\widehat{y}_t - \widehat{N}_t\right) = \eta_{kk}\left(\widehat{y}_{t-1} - \widehat{N}_{t-1}\right), \tag{30}$$

where η_{kk} is given in (24).

As the percentage deviation of output per efficiency unit of labor is a first-order difference equation, its rate of convergence is negatively related to η_{kk}, the coefficient in (30). When η_{kk} increases, the rate of convergence becomes slower. Another way to see the convergence speed is to use the concept of half-life (T), which is defined as the time required for any initial deviation from the balanced growth path to close one half of that deviation. In the current context of log-linearization, the time T required for closing half of the initial gap is defined by $\left(\widehat{y}_T - \widehat{N}_T\right) = 0.5\left(\widehat{y}_0 - \widehat{N}_0\right) = \eta_{kk}^T\left(\widehat{y}_0 - \widehat{N}_0\right)$, or equivalently

$$T = \frac{\ln 0.5}{\ln \eta_{kk}}. \tag{31}$$

Since $0 < \eta_{kk} < 1$ and $\ln \eta_{kk}$ is negative, a higher η_{kk} leads to a higher half-life according to (31), which is an alternative way to represent a slower convergence rate.

To study how η_{kk} varies with α, we use (24) to obtain

$$\frac{\partial \eta_{kk}}{\partial \alpha} = \frac{1}{2Q_1}\left\{\frac{\partial(Q_1 Q_2)}{\partial \alpha} + \frac{\partial(Q_3 Q_4)}{\partial \alpha} - \frac{(1 + Q_1 Q_2 + Q_3 Q_4)\left[\frac{\partial(Q_1 Q_2)}{\partial \alpha} + \frac{\partial(Q_3 Q_4)}{\partial \alpha}\right] - 2\frac{\partial(Q_1 Q_2)}{\partial \alpha}}{\sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4Q_1 Q_2}}\right\}$$

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[8]For the deterministic version of the baseline RBC model considered in this paper, (3), (11), (12) and (23) become $\widehat{y}_t = \alpha \widehat{N}_t + (1-\alpha)\widehat{k}_t$, $\widehat{N}_t = \lambda_{nk}\widehat{k}_t - \lambda_{nc}\widehat{c}_t$, $\widehat{k}_{t+1} = \lambda_{kk}\widehat{k}_t + \lambda_{kn}\widehat{N}_t - \lambda_{kc}\widehat{c}_t$, and $\widehat{k}_{t+1} = \eta_{kk}\widehat{k}_t$, respectively. Combining them, we obtain $\widehat{N}_t = \xi\widehat{k}_t$, where $\xi = \frac{\lambda_{nc}(\eta_{kk} - \lambda_{kk}) + \lambda_{nk}\lambda_{kc}}{\lambda_{kc} + \lambda_{kn}\lambda_{nc}}$. Therefore, we have

$$\widehat{N}_t = \xi\widehat{k}_t = \xi\eta_{kk}\widehat{k}_{t-1} = \eta_{kk}\widehat{N}_{t-1}. \tag{Δ}$$

Similarly, we obtain

$$\widehat{y}_t = \alpha\widehat{N}_t + (1-\alpha)\widehat{k}_t = (1 - \alpha + \alpha\xi)\widehat{k}_t = (1 - \alpha + \alpha\xi)\eta_{kk}\widehat{k}_{t-1} = \eta_{kk}\widehat{y}_{t-1}. \tag{$\Delta\Delta$}$$

Combining (Δ) and ($\Delta\Delta$) gives (30).

$$-\frac{1}{2\left(Q_1\right)^2}\left[\left(1+Q_1Q_2+Q_3Q_4\right)-\sqrt{\left(1+Q_1Q_2+Q_3Q_4\right)^2-4Q_1Q_2}\right]\frac{\partial Q_1}{\partial \alpha}. \qquad (32)$$

After performing some intricate algebra, we show in the Appendix that for the baseline RBC model,[9]

$$\frac{\partial \eta_{kk}}{\partial \alpha} < 0. \qquad (32a)$$

This result is also confirmed by the numerical results in Table 1. For the deterministic version of the baseline RBC model, when α increases, diminishing marginal returns set in more quickly. As a result, η_{kk} decreases and the rate of convergence increases.

5. CONCLUSION

It cannot be denied that the RBC approach has firmly established itself as a workhorse for macroeconomic analysis. However, while it has been widely used in the research community, learners may not readily grasp the underlying mechanism since this literature is usually quantitative and computational in nature. To make the underlying mechanism of the RBC models more readily comprehensible, Campbell (1994) employs an analytical approach and forcefully demonstrates its usefulness.

In this paper, we explore further the potential of this analytical approach. One important observation we make is that although the loglinear approximate equations are complicated, they are not excessively so. Because of the relationships among the steady-state values of various variables, the coefficients in the loglinear approximate equations are related to the underlying preference and technology parameters in a relatively simple manner. By exploiting this relationship, important economic concepts

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[9]As explained in footnote 5, β and θ are chosen to match the steady-state values of real interest rate and leisure, respectively. Therefore, $\frac{\partial \eta_{kk}}{\partial \alpha}$ in (32) should be interpreted as differentiating η_{kk} with respect to α, holding constant γ, g, δ, and the steady-state values of real interest rate and leisure.

(such as η_{kk} in the model considered in this paper) can be lucidly expressed in terms of the underlying parameters. As such, the presentation of the closed-form approximate solution in this paper is, arguably, neater than that of Campbell (1994). Also, the choice of the quadratic roots (for η_{kk}) and the demonstration of the uniqueness of the solution are now more transparent. In the same vein, comparative static results can, in principle, be confirmed analytically. By straightforward differentiation, we supply an illustration by proving a monotonic relationship between the rate of convergence of output and the Cobb-Douglas production function parameter.

We obtain the above conclusions based on a baseline RBC model. It is likely that most of these conclusions can be generalized to more complicated models. For a more complicated RBC model, while the effort in obtaining an approximate analytical solution may be more demanding, we believe that the results obtained would justify the effort spent, as in the model considered in this paper. It is hoped that such demonstrations will remove the "mystery" formerly alleged to enshroud the RBC approach, and encourage the reader to make greater use of it.

APPENDIX

(A) Loglinear approximation of the nonlinear system:

After simplification, (8a) becomes

$$A\left(\frac{N}{k}\right)^{\alpha} = \frac{1}{1-\alpha}\left[\frac{1+g}{\beta} - (1-\delta)\right]. \tag{A1}$$

Simplifying (7a) and using (A1) leads to

$$c = \left[A\left(\frac{N}{k}\right)^{\alpha} - (\delta+g)\right]k = \left[\frac{1+g-\beta(1-\delta)}{\beta(1-\alpha)} - (\delta+g)\right]k. \tag{A2}$$

Loglinearizing (9) gives

$$\theta(1-N)^{-\gamma}c\left[\widehat{c}_t + \gamma\left(\frac{N}{1-N}\right)\widehat{N}_t\right] = \alpha A\left(\frac{k}{N}\right)^{1-\alpha}\left[\widehat{A}_t + (1-\alpha)\widehat{k}_t - (1-\alpha)\widehat{N}_t\right]. \tag{A3}$$

Substituting (9a) into (A3) and simplifying, we obtain (11) with (14) and (15).

Loglinearizing (7) and using (A2) gives

$$(1+g)\, k\widehat{k}_{t+1} = \left[1 - \delta + (1-\alpha)\, AN^\alpha k^{-\alpha}\right] k\widehat{k}_t + AN^\alpha k^{1-\alpha}\widehat{A}_t$$

$$+ \alpha AN^\alpha k^{1-\alpha}\widehat{N}_t - \left[\frac{1+g-\beta(1-\delta)}{\beta\,(1-\alpha)} - (\delta+g)\right] k\widehat{c}_t. \tag{A4}$$

Using (A1) and (22) to simplify (A4), we obtain (12) with (16), (17), (18) and (19).

Loglinearizing (8) and simplifying leads to

$$\widehat{c}_t = \frac{\beta}{1+g}E_t\left\{\left[1-\delta + (1-\alpha)\, A\left(\frac{N}{k}\right)^\alpha\right]\widehat{c}_{t+1} - (1-\alpha)\, A\left(\frac{N}{k}\right)^\alpha\left(\widehat{A}_{t+1} + \alpha\widehat{N}_{t+1} - \alpha\widehat{k}_{t+1}\right)\right\}. \tag{A5}$$

Substituting (A1) and (22) into (A5) and simplifying, we obtain (13) with (20) and (21).

(B) Solution to the loglinear system of expectational equations:

Equation (4) can be written as $\widehat{A}_t = \phi\widehat{A}_{t-1} + \varepsilon_t$. Therefore,

$$E_t\widehat{A}_{t+1} = \phi\widehat{A}_t, \tag{A6}$$

Substituting (11) into (12) and rearranging leads to

$$\widehat{c}_t = \frac{(\lambda_{kk} + \lambda_{kn}\lambda_{nk})\widehat{k}_t + (\lambda_{ka} + \lambda_{kn}\lambda_{na})\widehat{A}_t - \widehat{k}_{t+1}}{\lambda_{kc} + \lambda_{kn}\lambda_{nc}}. \tag{A7}$$

Leading (A7) one period and taking expectation, we obtain

$$E_t\widehat{c}_{t+1} = \frac{(\lambda_{kk} + \lambda_{kn}\lambda_{nk})\widehat{k}_{t+1} + (\lambda_{ka} + \lambda_{kn}\lambda_{na})E_t\widehat{A}_{t+1} - E_t\widehat{k}_{t+2}}{\lambda_{kc} + \lambda_{kn}\lambda_{nc}}. \tag{A8}$$

We eliminate $\widehat{N}_{t+1}$ in (13) by leading (11) one period and then substituting it into (13). We then substitute (A6) to (A8) into this expression to obtain the following equation in terms of capital stock and technology shock only:

$$\frac{1}{\lambda_{kc} + \lambda_{kn}\lambda_{nc}}\left[(\lambda_{kk} + \lambda_{kn}\lambda_{nk})\widehat{k}_t + (\lambda_{ka} + \lambda_{kn}\lambda_{na})\widehat{A}_t - \widehat{k}_{t+1}\right]$$

$$= \frac{1 + \lambda_{cn}\lambda_{nc}}{\lambda_{kc} + \lambda_{kn}\lambda_{nc}} \left[(\lambda_{kk} + \lambda_{kn}\lambda_{nk})\widehat{k}_{t+1} + (\lambda_{ka} + \lambda_{kn}\lambda_{na})\phi\widehat{A}_t - E_t\widehat{k}_{t+2} \right]$$

$$+ (\lambda_{ck} - \lambda_{cn}\lambda_{nk})\widehat{k}_{t+1} - (\lambda_{ca} + \lambda_{cn}\lambda_{na})\phi\widehat{A}_t \qquad (A9)$$

From (23) and (4), we obtain $E_t\widehat{k}_{t+2} = \eta_{kk}^2 \widehat{k}_t + \eta_{ka}(\eta_{kk} + \phi)\widehat{A}_t$. Substituting this expression and (23) into (A9) leads to

$$(\lambda_{kk} + \lambda_{kn}\lambda_{nk})\widehat{k}_t + (\lambda_{ka} + \lambda_{kn}\lambda_{na})\widehat{A}_t - (\eta_{kk}\widehat{k}_t + \eta_{ka}\widehat{A}_t) = (1 + \lambda_{cn}\lambda_{nc})$$

$$\left\{ (\lambda_{kk} + \lambda_{kn}\lambda_{nk})(\eta_{kk}\widehat{k}_t + \eta_{ka}\widehat{A}_t) + (\lambda_{ka} + \lambda_{kn}\lambda_{na})\phi\widehat{A}_t - \left[\eta_{kk}^2\widehat{k}_t + \eta_{ka}(\eta_{kk} + \phi)\widehat{A}_t \right] \right\}$$

$$+ (\lambda_{kc} + \lambda_{kn}\lambda_{nc}) \left[(\lambda_{ck} - \lambda_{cn}\lambda_{nk})\left(\eta_{kk}\widehat{k}_t + \eta_{ka}\widehat{A}_t \right) - (\lambda_{ca} + \lambda_{cn}\lambda_{na})\phi\widehat{A}_t \right]. \qquad (A10)$$

Equating the coefficients on $\widehat{k}_t$ of (A10) leads to the following quadratic equation in η_{kk}:

$$Q_1\eta_{kk}^2 - (1 + Q_1Q_2 + Q_3Q_4)\eta_{kk} + Q_2 = 0, \qquad (A11)$$

where the "Q" coefficients, which are complicated functions of the deep parameters, are given in (25) to (28). It can be shown that the determinant of the quadratic equation (A11),

$$(1 + Q_1Q_2 + Q_3Q_4)^2 - 4Q_1Q_2 = (Q_1Q_2 + Q_3Q_4 - 1)^2 + 4Q_3Q_4 > 0. \qquad (A12)$$

Therefore, the two roots to the quadratic equation (A11) are real and unequal, and they are given by (24) and

$$\eta_2 = \frac{(1 + Q_1Q_2 + Q_3Q_4) + \sqrt{(1 + Q_1Q_2 + Q_3Q_4)^2 - 4Q_1Q_2}}{2Q_1}. \qquad (A13)$$

It can further be shown that

$$0 < \eta_{kk} < 1 < \frac{1}{\beta} < \eta_2. \qquad (A14)$$

The larger root, η_2, is therefore excluded since it is inconsistent with the transversality condition (10).

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A sketch of the proof of (A14) is as follows. It is based on the inequalities in (25) to (28). First, since all Qs are positive, we have

$$1 + Q_1 Q_2 + Q_3 Q_4 > \sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2}. \qquad \text{(A15)}$$

Therefore, it is easy to conclude from (24) that $\eta_{kk} > 0$. Second, from (25) to (28), we obtain

$$\left[(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2 \right] - \left[(1 + Q_1 Q_2 + Q_3 Q_4) - 2 Q_1 \right]^2$$

$$= 4 Q_1 \left[(Q_1 - 1)(Q_2 - 1) + Q_3 Q_4 \right] > 0.$$

Therefore,

$$\sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2} > (1 + Q_1 Q_2 + Q_3 Q_4) - 2 Q_1,$$

and it can be seen from (24) that $\eta_{kk} < 1$. Finally, we know from (A12), (A13), (16) and (25) to (28) that

$$\eta_2 = \frac{(1 + Q_1 Q_2 + Q_3 Q_4) + \sqrt{(Q_1 Q_2 + Q_3 Q_4 - 1)^2 + 4 Q_3 Q_4}}{2 Q_1}$$

$$> \frac{(1 + Q_1 Q_2 + Q_3 Q_4) + (Q_1 Q_2 + Q_3 Q_4 - 1)}{2 Q_1} > Q_2 > \frac{1}{\beta}.$$

(C) Monotonicity of η_{kk} with respect to α:

The first term on the right-hand side of (32) can be rewritten as

$$\frac{1}{2 Q_1} \left\{ \left(Q_1 \frac{\partial Q_2}{\partial \alpha} + Q_2 \frac{\partial Q_1}{\partial \alpha} + Q_3 \frac{\partial Q_4}{\partial \alpha} + Q_4 \frac{\partial Q_3}{\partial \alpha} \right) \left[1 - \frac{(1 + Q_1 Q_2 + Q_3 Q_4)}{\sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2}} \right] \right\}$$

$$- \frac{1}{Q_1} \left[\frac{\left(Q_1 \frac{\partial Q_2}{\partial \alpha} + Q_2 \frac{\partial Q_1}{\partial \alpha} \right)}{\sqrt{(1 + Q_1 Q_2 + Q_3 Q_4)^2 - 4 Q_1 Q_2}} \right] \qquad \text{(A16)}$$

It can be shown from (14), (18) and (19) that

$$\frac{\partial \lambda_{nc}}{\partial \alpha} = \left[1 - \alpha + \gamma \left(\frac{N}{1 - N} \right) \right]^{-2} > 0, \qquad \text{(A17a)}$$

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and

$$\frac{\partial \lambda_{kn}}{\partial \alpha} = \frac{\partial \lambda_{kc}}{\partial \alpha} = \frac{1 - \beta(1 - \delta_g)}{\beta(1 - \alpha)^2} > 0. \qquad \text{(A17b)}$$

Therefore, we obtain from (17a), (17b), and (25) to (28) that

$$\frac{\partial Q_1}{\partial \alpha} = \frac{[1 - \beta(1 - \delta_g)]\left[1 + \gamma\left(\frac{N}{1-N}\right)\right]}{\left[1 - \alpha + \gamma\left(\frac{N}{1-N}\right)\right]^2} > 0, \qquad \text{(A18a)}$$

$$\frac{\partial Q_2}{\partial \alpha} = \frac{[1 - \beta(1 - \delta_g)]\left[1 + \gamma\left(\frac{N}{1-N}\right)\right]}{\beta\left[1 - \alpha + \gamma\left(\frac{N}{1-N}\right)\right]^2} > 0, \qquad \text{(A18b)}$$

$$\frac{\partial Q_3}{\partial \alpha} = \frac{\partial \lambda_{kc}}{\partial \alpha} + \lambda_{kn}\frac{\partial \lambda_{nc}}{\partial \alpha} + \lambda_{nc}\frac{\partial \lambda_{kn}}{\partial \alpha} > 0, \qquad \text{(A18c)}$$

and

$$\frac{\partial Q_4}{\partial \alpha} = \frac{[1 - \beta(1 - \delta_g)]\gamma\left(\frac{N}{1-N}\right)\left[1 + \gamma\left(\frac{N}{1-N}\right)\right]}{\left[1 - \alpha + \gamma\left(\frac{N}{1-N}\right)\right]^2} > 0. \qquad \text{(A18d)}$$

Using (25) to (28), (A12), (A15), (A16), and (A18a) to (A18d), we conclude from (32) that $\frac{\partial \eta_{kk}}{\partial \alpha} < 0$.

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Table 1. Numerical values of η_{kk} and the half-life for the baseline RBC model

α	γ				
	0.001	0.2	1	5	1000
0.2	0.9902	0.9903	0.9907	0.9912	0.9915
	70.45	71.46	74.15	78.31	81.37
0.33	0.9809	0.9814	0.9825	0.9839	0.9848
	35.93	36.83	39.20	42.79	45.37
0.58	0.9482	0.9511	0.9570	0.9633	0.9667
	13.04	13.83	15.79	18.55	20.44
0.67	0.9257	0.9312	0.9417	0.9517	0.9567
	8.98	9.73	11.54	14.01	15.65

Note: The top number in each cell is η_{kk} according to (24), and the bottom number is the half-life (in quarters) according to (31). Parameter γ is the reciprocal of the elasticity of intertemporal substitution for leisure, and α is the exponent on labor in the Cobb-Douglas production function. Parameter α is set at values commonly used in other papers (such as Lau, 2002), and parameter γ is set at different values to cover a wide range of possibilities. The assumed values of parameters δ and g are 0.025 and 0.005 respectively. The steady-state value of leisure is 1/3. To make the steady-state real interest rate equal to 0.015, the implied value of β is 0.990.