A Market Approach to the Bankruptcy Problem[*]

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Abstract

This paper proposes a new market-based approach to the bankruptcy problem. A hypothetical claims market, where creditors can trade their claims prior to the allocation of the liquidation value, is considered. The introduction of the claims market opens an atypical arbitrage opportunity, which allows creditors to make profits even by buying and selling the same amount of claims at the same price. This anomaly occurs because the claims market enables the creditors to exploit any sensitivity of a bankruptcy rule to a change in the distribution of claims, and is ruled out by requiring that the bankruptcy rule satisfy a *no-arbitrage* condition. The no-arbitrage condition turns out to be a necessary and sufficient condition for the existence of equilibrium in the claims market. All equilibria are shown to be equivalent to the outcome of the proportional rule. A connection between the market-based approach and the axiomatic approach is developed and simpler characterizations of the proportional rule are derived. A new normative foundation for the proportional rule is also established.

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1. Introduction

This paper considers the problem of distributing the liquidation value of a bankrupt firm amongst its various creditors in the same precedence class. This problem is known as the *bankruptcy problem* and has been extensively studied in the literature on cost allocation. The bankruptcy problem is important because it provides a unified framework for studying a wide variety of allocation problems, ranging from income taxation to estate division.

The existing literature has analyzed the bankruptcy problem from either an axiomatic perspective [e.g., Moulin (1985, 1987), Young (1987, 1988), Chun (1988)] or a game-theoretic one [e.g., O'Neill (1982), Aumann and Maschler (1985)]. The former approach characterizes and compares various bankruptcy rules[1] in terms of intuitively appealing axioms. The latter approach typically associates with each bankruptcy problem either a corresponding bargaining problem or a coalitional-form game, and solution concepts developed in cooperative game theory then give rise to different bankruptcy rules. The reader is referred to Thomson (1995) for a comprehensive survey of the literature.

This paper approaches the bankruptcy problem from a "market-based" perspective. Given the fundamental role of the market-mechanism in resource allocation, it is somewhat surprising that a market-based approach to the bankruptcy problem seems never to have been proposed. It is therefore of interest to study the allocation of the liquidation value using the market-mechanism.[2]

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[1] A *bankruptcy rule* assigns a unique allocation to every bankruptcy problem. A formal definition is provided in Section 2.

[2] The case for a market-based approach could also be motivated by arguing that creating such a market could be potentially welfare enhancing. For example, in the context of Chapter 7 bankruptcy, a market would allow risk-averse creditors to unload the uncertainties arising from liquidation. Such specific motivations are not pursued here, as they would change the nature of the bankruptcy problem fundamentally.

We begin by introducing a "claims market" in which creditors can trade their claims prior to the allocation of the liquidation value amongst them. In the bankruptcy problem, the claims market results in a unique two-stage market-rule structure. In the first (market) stage, agents are allowed to trade their claims in a market. In the second (liquidation) stage, the liquidation value is divided among the claimants according to some bankruptcy rule, based on the post-trade distribution of claims. We then ask the following question: "In the class of all permissible bankruptcy rules to be used in the liquidation stage, which rules (if any) are consistent with an equilibrium in the claims market?" In other words, given that the agents know which rule will be used for allocation in the second stage, will there exist an equilibrium in the claims market? If so, is it possible to characterize such bankruptcy rules, and determine the equilibrium outcome of the claims market?

As we will elaborate later, an important feature of the market-rule structure is that it potentially allows agents to make profits even by buying and selling the *same* amount of claims at the *same* price. Intuitively speaking, this anomaly occurs because the creditors can use the claims market to change the distribution of claims at zero net cost, and this activity can be profitable if the bankruptcy rule is sensitive to the distribution of claims. We rule out such arbitrage opportunities by imposing a *no-arbitrage* condition, which requires that the bankruptcy rule be invariant to the distribution of claims.

It turns out that no-arbitrage is more than an intuitively appealing property. We show that it is a necessary and sufficient condition for the existence of equilibrium in the claims market. That is, only rules satisfying the no-arbitrage condition are consistent with equilibrium in the claims market. Moreover, it allows us to derive demand and supply curves of an individual's claims independently from other individuals. It is shown that the claims market equilibrium is unique[3], and that the market value of each creditor's claim coincides with the allocation according to the *proportional rule*.

It is also of interest to explore the relationship between the market-based approach and the existing axiomatic approach, as this helps develop a unified methodology of studying the bankruptcy problem. The no-arbitrage condition provides a basis for exploring the connection with previous work on the bankruptcy problem. In particular, it provides a natural interpretation of the *decentralizability* axiom [Moulin (1988)] that has been used to characterize the proportional rule in the axiomatic approach. Moreover, the no-arbitrage condition has intuitively appealing normative implications.

The rest of the paper is organized as follows. The next section sets up the hypothetical claims market and studies its equilibria. Section 3.1 discusses the connection between the no-arbitrage condition and certain axioms with a strategic flavor. This exercise results in simpler and cleaner characterizations of the proportional rule. Section 3.2 uses the no-arbitrage condition to deduce a normative implication of the proportional rule. It also presents a new characterization of the proportional rule based on this normative implication. Section 4 contains some concluding remarks.

2. A Market for Claims

We begin by formally defining the bankruptcy problem. Let $A \geq 0$ be the liquidation value of the bankrupt firm. The set of all creditors is denoted by N and the claim of a creditor $i \in N$ by $c_i \geq 0$. Let $c = (c_i)_{i \in N}$ be a profile of claims. A **bankruptcy problem** is a pair

[3] To be precise, the equilibrium price of claims is unique but not the equilibrium quantity of claims traded.

$(c;A)$ such that $\sum c_i \geq A$. Let B^n be the collection of all bankruptcy problems, where n is the number of creditors in N.

It is assumed that the division of the liquidated value is governed by a systematic procedure, or a **bankruptcy rule**, which is formulated as a function $f : \mathrm{B}^n \to \Re^n_+$ such that for any problem $(c;A)$,

- For any creditor i, if $c_i = 0$, then $f_i(c;A) = 0$ (**null-player**);
- $\sum f_i(c;A) = A$ (**efficiency**).

Null-player says that if the claim of a creditor is zero, then this creditor receives nothing; efficiency requires that the liquidated value be entirely distributed to the creditors.

Suppose that there exists a market for claims, which gives the creditors the option to trade their claims with each other prior to receiving their shares of the liquidated value. Now creditor i can buy $c^b \in \Re^{n-1}_+$ at price $p^b \in \Re^{n-1}_+$ from the market, and sell $c^s \in \Re^{n-1}_+$ at price $p^s \in \Re^{n-1}_+$ to the market, before the firm is liquidated. These transactions will change the distribution of claims from c to c', e.g. $c'_i = c_i + \sum_{j \neq i} c^b_j - \sum_{j \neq i} c^s_j$. Agent i can make a profit from these transactions if:

(1) $$f_i(c_i, c_{-i}; A) - p^b \cdot c^b + p^s \cdot c^s < f_i(c'_i, c'_{-i}; A).$$

Consider the special case where the amount bought by agent i is equal to the amount sold and the buying and selling prices are the same, i.e. $\sum_{j \neq i} c^b_j = \sum_{j \neq i} c^s_j$ and $p^b_j = p^s_k$ for all $j, k \neq i$. From equation (1), agent i will still be able to earn an "arbitrage" profit if $f_i(c_i, c_{-i}; A) < f_i(c'_i, c'_{-i}; A)$. This profit arises from exploiting any distribution sensitivity of the bankruptcy rule in the liquidation stage, via innocuous market transactions.[4] To rule out this anomaly, we impose the following condition on a bankruptcy rule.

No-(liquidation-)arbitrage:[5] For any agent i and any two problems $(c_i, c_{-i}; A)$ and $(c_i, c'_{-i}; A)$, if $\sum_{j \neq i} c_j = \sum_{j \neq i} c'_j$, then $f_i(c_i, c_{-i}; A) = f_i(c_i, c'_{-i}; A)$.

This condition is interpreted intuitively as *decentralizability* by Moulin (1988).[6] The no-arbitrage condition is vacuous when there are less than three agents. It is assumed that, unless specified otherwise, there are at least three agents.

A serious problem arises if the no-arbitrage condition is not imposed -- the claims market equilibria fail to exist (Proposition 1). Given a bankruptcy problem $(c; A)$, an equilibrium of the claims market is a pair (c^*, p^*) such that

- $f_i(c_i^*, c_{-i}^*; A) \geq f_i(c_i', c_{-i}'; A) + p^*(c_i^* - c_i')$ for all i and all c' s.t. $\sum c_j' = \sum c_j$ and
- $\sum c_j^* = \sum c_j$.

The first condition says that no single agent can benefit from deviating from the equilibrium vector of claims c^* through trading claims at price p^*, and the second condition is simply market clearing.

Proposition 1: *For any bankruptcy rule f, if an equilibrium exists in the claims market, then f satisfies the no-arbitrage condition.*

[4] It is even possible for an agent to use a *"buy high, sell low"* strategy to earn an arbitrage profit when the rule is sufficiently sensitive to the distribution of the claims and the price differential is small enough.

[5] The term "liquidation" is used to indicate that we are confining our attention to those arbitrage activities where the arbitrageur potentially makes a profit only in the second (liquidation) stage of the game. Clearly, buying and selling claims in equal amounts and at the same price cannot yield any profit in the first (market) stage.

[6] The discussion of the related literature is postponed to the next section.

Proof: Consider a bankruptcy problem $(c;A)$. Suppose that *f* is a bankruptcy rule for which an equilibrium exists in the claims market. Let (c^*,p^*) be an equilibrium and c' be such that $\sum c_i' = \sum c_i$. Summing-up the first equilibrium condition over all the agents yields

(2) $$\sum f_i(c_i^*,c_{-i}^*;A) \geq \sum f_i(c_i',c_{-i}';A) + p^* \sum (c_i^* - c_i').$$

Market clearing implies that the second term of the RHS of Equation (2) equals zero. It then follows from efficiency that

(3) $$\sum f_i(c_i^*,c_{-i}^*;A) = A = \sum f_i(c_i',c_{-i}';A).$$

Equation (3) implies that Equation (2) should hold with equality, because if it holds with strict inequality for an agent *j*, then summing over all agents will result in a violation of Equation (3). Since c' is arbitrary, Equation (2) holds (with equality) for all $c'' = (c_i', c_{-i}'')$ such that $\sum_{j \neq i} c_j'' = \sum_{j \neq i} c_j'$. Hence,

(4) $$f_i(c_i',c_{-i}';A) = f_i(c_i',c_{-i}'';A).$$

Having established the necessity of the no-arbitrage condition for the existence of equilibrium, we now proceed to ask the following questions: Is the no-arbitrage condition sufficient to guarantee the existence of equilibrium in the claims market? If so, is the equilibrium unique? What is (are) the equilibrium claims-vector(s) and price(s)? To answer all these questions, we construct and use the market demand and supply curves to identify the whole set of equilibria. Let $(c;A)$ be the underlying bankruptcy problem, so that *c* denotes the creditors' claims prior to any trade. Let c' denote the claims of the creditors after the trade. Since the total amount of claims available in the market is constrained by the underlying bankruptcy problem $(c;A)$, $\sum c_j' = \sum c_j$.

We begin by determining the total value of holding a claim c_i' by an agent i. The total value depends on what c_i' entitles agent i to receive from the bankruptcy rule, i.e. $f_i(c_i', c_{-i}'; A)$.

Proposition 2: *Under no-arbitrage, the total value of holding a claim* $c_i' \in [0, C]$ *by an agent* *i is* $\frac{A}{C} c_i'$, where $C = \sum c_j$.

Proof: First, null-player implies that $f_i(c_i', c_{-i}'; A) = 0$ if $c_i' = 0$ and efficiency implies that $f_i(c_i', c_{-i}'; A) = A$ if $c_i' = C$. The next case is $c_i' = \frac{kC}{l}$, where k and l are any positive integers such that $k \le l$. We start with the profile $(\frac{C}{l}, \ldots, 0, \ldots, \frac{(l-1)C}{l})$ and construct a double-arbitrage by agents n and i as follows:

- agent n buys $\frac{C}{l}$ from agent 1 and resells it to agent i;
- agent i buys $\frac{C}{l}$ from agent n and resells it to agent 1.

This double-arbitrage may also be viewed as a single-arbitrage by agent 1, who buys $\frac{C}{l}$ from agent n and resells it to agent i. These arbitrages are depicted in Figure 1.

Figure 1: *The 1st Double-Arbitrage and Single-Arbitrage*

$$(\frac{C}{l}, \ldots, 0, \ldots, \frac{(l-1)C}{l}) \xrightarrow{n \text{ arbitrages}} (0, \ldots, \frac{C}{l}, \ldots, \frac{(l-1)C}{l}) \xrightarrow{i \text{ arbitrages}} (\frac{C}{l}, \ldots, \frac{C}{l}, \ldots, \frac{(l-2)C}{l})$$

$$\xrightarrow[\text{1 arbitrages}]{}$$

The claims of agents 2, $\ldots, i-1,\ i+1, \ldots,\ n-1$ are always zero. Null-player implies that they never receive anything. The allocation that corresponds to the first profile in the first double-arbitrage can be determined by the following three steps:

1. Let the amount agent 1 receives be x_1.
2. Null-player implies that agent i receives nothing.
3. Efficiency implies that agent n receives $A-x_1$.

(5) $$(x_1,\ldots,0,\ldots,A-x_1) = f(\frac{C}{l},\ldots,0,\ldots,\frac{(l-1)C}{l};A).$$

A similar argument can be used to determine the allocation that corresponds to the second profile:

1. Null-player implies that agent 1 receives nothing.
2. No-arbitrage implies that agent n receives $A-x_1$.
3. Efficiency implies that agent i receives x_1.

(6) $$(0,\ldots,x_1,\ldots,A-x_1) = f(0,\ldots,\frac{C}{l},\ldots,\frac{(l-1)C}{l};A).$$

Similarly, the allocation that corresponds to the third profile follows from:

1. No-arbitrage implies that agent i receives x_1 and agent 1 receives x_1 (the amount agent 1 receives is deduced from the arbitrage between the 1st and 3rd profiles).
2. Efficiency implies that agent n receives $A-2x_1$.

(7) $$(x_1,\ldots,x_1,\ldots,A-2x_1) = f(\frac{C}{l},\ldots,\frac{C}{l},\ldots,\frac{(l-2)C}{l};A).$$

The same argument can be used to determine the allocations in the second double-arbitrage, which are shown in Figure 2.

Figure 2: *The 2nd Double-Arbitrage and the Corresponding Allocations*

$$(\frac{C}{l},\ldots,\frac{C}{l},\ldots,\frac{(l-2)C}{l}) \xrightarrow{n \text{ arbitrages}} (0,\ldots,\frac{2C}{l},\ldots,\frac{(l-2)C}{l}) \xrightarrow{i \text{ arbitrages}} (\frac{C}{l},\ldots,\frac{2C}{l},\ldots,\frac{(l-3)C}{l})$$

$$\xrightarrow{\text{l arbitrages}}$$

$$\ldots$$

$$(x_1,\ldots,x_1,\ldots,A-2x_1) \rightarrow (0,\ldots,2x_1,\ldots,A-2x_1) \rightarrow (x_1,\ldots,2x_1,\ldots,A-3x_1)$$

Similarly, the allocation that corresponds to the middle profile in the k-th double-arbitrage is

(8) $$(0,\ldots,kx_1,\ldots,A-kx_1) = f(0,\ldots,\frac{kC}{l},\ldots,\frac{(l-k)C}{l};A).$$

If $k=l$, it follows from the first case that $lx_1 = A$. Hence, $f_i(c_i', c_{-i}'; A) = \frac{kA}{l}$ if $c_i' = \frac{kC}{l}$.

The remaining case is $c_i' \neq \frac{kC}{l}$ for any positive integers $k \leq l$. It is sufficient to show that f_i is "monotonic in its own claim"[7], i.e. $f_i(c_i'', c_{-i}''; A) \geq f_i(c_i', c_{-i}'; A)$ if $c_i'' \geq c_i'$ and $\sum c_j'' = C$. Without loss of generality, we can assume that there is an agent $j \neq i$ such that $c_j'' = 0$ and $c_j' = c_i'' - c_i'$ and the claims of all other agents are the same across the two profiles c' and c''. Then no-arbitrage implies that

(9) $$f_i(c_i'', c_{-i}''; A) = f_i(c_i', c_{-i}'; A) + f_j(c_i', c_{-i}'; A) \geq f_i(c_i', c_{-i}'; A).$$

Since agent i was chosen arbitrarily, Proposition 2 holds for all agents. Hence, the marginal value of holding a dollar of claim is always $\frac{A}{C}$. The demand and supply curves are described in Equation (10).

Corollary 1: T*he demand correspondence* $D: \Re_+ \rightarrow\rightarrow \Re_+$ *and supply correspondence* $S: \Re_+ \rightarrow\rightarrow \Re_+$ *of claims are as follows:*

[7] This is not a regular monotonicity condition, since an increase in agent i's claim is associated with an equal overall reduction in other claims.

(10) $$D(p)=\begin{cases}0 & if\ p>\dfrac{A}{C}\\ [0,C] & if\ p=\dfrac{A}{C}\\ C & if\ p<\dfrac{A}{C}\end{cases}\quad \text{and} \quad S(p)=\begin{cases}0 & if\ p<\dfrac{A}{C}\\ [0,C] & if\ p=\dfrac{A}{C}\\ C & if\ p>\dfrac{A}{C}\end{cases},$$

where p is the price of a claim.

Remark 1: Corollary 1 confirms Moulin's intuition. No-arbitrage, or decentralizability, allows us to construct the demand curve of an individual independently from others.

Since the total quantity of claims is fixed at C, the aggregate (or market) demand and supply curves are the same as the individual ones. It is clear that $\frac{A}{C}$ is the unique equilibrium market price. The quantity is indeterminate, since the creditors are indifferent between holding their claims and selling them at the equilibrium price. They receive the same (final) amount either way. Our results thus far are summarized in the following theorem:

Theorem 1:

a. *An equilibrium exists in the claims market if, and only if, the no-arbitrage condition is satisfied.*

b. *The equilibrium market value of creditor i's claim* c_i *is* $\frac{A}{C}c_i$, *which is the same as the proportional outcome.*

The central position of the no-arbitrage condition in the analysis of the claims market has been established. It has been shown to be both necessary and sufficient for the existence of equilibrium, and has also been used to characterize the whole set of equilibria.

3. Axiomatic Approach

To connect the results in the previous section to the ones in the axiomatic literature, an explicit discussion of the claims market is suppressed in this section, and the no-arbitrage condition is now viewed as an axiom. For ease of exposition, our analysis is divided into subsections – the first focusing on axioms with a "strategic" flavor and the second on axioms with a "normative" flavor. The term "strategic" is used to connote axioms that involve manipulation of the bankruptcy rule by agents in order to maximize their reward. By contrast, the term "normative" is reserved for axioms that are viewed as embodying some commonly accepted notion of fairness, such as anonymity or symmetry.

3.1 Strategic Analysis

We proceed by asking the following question: "Which bankruptcy rule is consistent with no-arbitrage?" The proof of Proposition 2 can be used (without any modification) to show that the proportional rule is the only rule that satisfies this property.

Corollary 2: *The proportional rule is the only rule that satisfies no-arbitrage (or decentralizability).*

There is a slight difference between Corollary 2 and Moulin's (1988) result. He requires $0 \le f_i(c;A) \le c_i$ (**core property**), while we use null-player, a weaker condition.

It is well known that (under efficiency) decentralizability, or no-arbitrage, is logically equivalent to *no-advantageous reallocation* used by Moulin (1987) and Chun (1988). This property prevents a group of agents to gain from a pure reallocation of claims among themselves.

No-advantageous reallocation: For any group of agents Q and any two problems $(c_Q, c_{-Q}; A)$ and $(c'_Q, c_{-Q}; A)$, if $\sum_{i\in Q} c_i = \sum_{i\in Q} c'_i$, then $\sum_{i\in Q} f_i(c_Q, c_{-Q}; A) = \sum_{i\in Q} f_i(c'_Q, c_{-Q}; A)$.

Corollary 3: *The proportional rule is the only rule that satisfies no-advantageous reallocation.*

Corollary 3 replaces "anonymity", "continuity" and dummy (defined below) in Chun's (1988) Theorem 2 by non-negativity and null-player (null-player is slightly weaker than dummy).

The connection can also be extended to another closely related property, known as *non-manipulability by merging and splitting*, used by O'Neill (1982) and Chun (1988). The fine distinction between the two is that non-manipulability prevents agents from gaining by merging and splitting.

Non-manipulability by merging and splitting: For any group of agents Q and any two problems $(c_Q, c_{-Q}; A)$ and $(c'_q, c_{-Q}; A)$, if $\sum_{i \in Q} c_i = c'_q$, then $\sum_{i \in Q} f_i(c_Q, c_{-Q}; A) = f_q(c'_q, c_{-Q}; A)$.[8]

The next corollary follows from noting that the proportional rule is non-manipulable and non-manipulability implies no-advantageous reallocation, as pointed out by Chun (1988).

Corollary 4: *The proportional rule is the only rule that satisfies non-manipulability by merging and splitting.*

This corollary can be strengthened by dropping null-player and extended to the two-agent case. To drop null-player, we show that efficiency and non-manipulability imply dummy. (Chun (1988) makes a similar observation, but he requires anonymity.)

Dummy: For any two problems $(c; A)$ and $(c, 0; A)$, $f(c, 0; A) = (f(c; A), 0)$.

Lemma 1: *Efficiency and non-manipulability by merging and splitting together imply dummy.*

Proof: Let $(c, 0; A)$ be any bankruptcy problem and $i \in N$ be any agent. For simplicity, assume that $n \geq 2$. If agents i and $n+1$ merge, non-manipulability implies that

[8] Banker (1981) strengthens this property by additionally requiring that the non-manipulating agents not be affected. It can be shown that the stronger version is implied by the original one and efficiency together.

$$f_i(c,0;A) + f_{n+1}(c,0;A) = f_i(c;A) \text{ for all } i \in N . \tag{11}$$

Summing Equation (11) over N,

$$\textstyle\sum_{i \in N} f_i(c,0;A) + nf_{n+1}(c,0;A) = \sum_{i \in N} f_i(c;A) . \tag{12}$$

Efficiency implies that

$$\textstyle\sum_{i \in N} f_i(c,0;A) + f_{n+1}(c,0;A) = A = \sum_{i \in N} f_i(c;A) . \tag{13}$$

Equations (12) and (13) imply that $f_{n+1}(c,0;A) = 0$, and Equation (11) implies that $f_i(c,0;A) = f_i(c;A)$ for all $i \in N$.

The extension to the two-agent case is straightforward. Non-manipulability implies that $f_i(c_i, c_j; A) = f_i(c_i, c'_j, c_j - c'_j; A) = \dfrac{c_i A}{c_i + c'_j + (c_j - c'_j)} = \dfrac{c_i A}{c_i + c_j}$.

The difference between Corollary 4 and Chun's (1988) Theorem 3 resembles the preceding comparison. In particular, we replace anonymity and continuity in Chun's Theorem 3 by non-negativity.[9]

3.2 Normative Analysis

The normative appeal of the proportional rule was totally suppressed by no-arbitrage in the previous subsection. Anonymity, the only normative property that has been considered so far, was replaced by more elementary properties. However, it would be incorrect to infer that the proportional rule is normatively unappealing.[10] This subsection establishes a normative foundation for the proportional rule. The normative axiom used is a "derivative" of no-arbitrage.

[9] In related work, de Frutos (1999) has also derived Corollary 4.

[10] Indeed, Aristotle used proportionality to *define* justice. "What is just …is what is proportional, and what is unjust is what violates proportion." (Aristotle, *Ethics*, Book V.)

Observant readers have probably deduced, from the first double-arbitrage in Section 2, that no-arbitrage implies that agents with equal claims receive the same amounts. This normative property is known as *symmetry*.

Symmetry: For any two agents i, j and any problem $(c;A)$, if $c_i = c_j$, then $f_i(c;A) = f_j(c;A)$.

Symmetry is a very mild property. One would expect that a stronger normative property could be derived from no-arbitrage. Indeed, no-arbitrage stretches the value of equal treatment to groups of agents. When the total claims of two groups of agents are the same, these two groups receive equal amounts.

Group symmetry: For any two groups Q, Q', and any problem $(c;A)$, if $\sum_{i\in Q} c_i = \sum_{j\in Q'} c_j$, then $\sum_{i\in Q} f_i(c;A) = \sum_{j\in Q'} f_j(c;A)$. [11]

Lemma 2: *If a rule satisfies no-arbitrage, then it is group symmetric.*

One could derive this lemma from Corollary 2. We prefer a direct proof that shows how these two properties are related.

Proof: We need to show that if the claims of any two groups Q, Q' in a problem $(c;A)$ are such that $\sum_{i\in Q} c_i = \sum_{j\in Q'} c_j$, then no-arbitrage implies that $\sum_{i\in Q} f_i(c;A) = \sum_{j\in Q'} f_j(c;A)$.

First, an agent $k \notin Q$ arbitrages by buying up the claims of group Q and reselling the whole amount to a single agent $q \in Q$. As a result, the profile c changes to $c' = (c'_Q, c_{-Q})$, where $c'_q = \sum_{i\in Q} c_i$ and $c'_i = 0$ for all $i \in Q/q$. No-arbitrages implies that $f_k(c';A) = f_k(c;A)$ for all $k \notin Q$. Null-player and efficiency imply that

(14) $$f_q(c';A) = A - \sum_{k\notin Q} f_j(c';A) = A - \sum_{k\notin Q} f_j(c;A) = \sum_{i\in Q} f_i(c;A).$$

[11] Notice that group symmetry is defined for two possibly *distinct* groups of agents, whereas no-advantageous reallocation is defined for the *same* group of agents.

Second, agent q arbitrages by buying up the claims of group Q' and reselling the whole amount to a single agent $q' \in Q'$. Consequently, the profile c' changes to $c'' = (c''_{Q'}, c'_{-Q'})$, where $c''_{q'} = \sum_{j \in Q'} c_j$ and $c''_j = 0$ for all $j \in Q'/q'$. No-arbitrage and Equation (14) imply that

(15) $$f_q(c''; A) = f_q(c'; A) = \sum_{i \in Q} f_i(c; A) .$$

This double-arbitrage is outlined in the upper part of Figure 3. The change from c to c'' can also be achieved by the alternative double arbitrage which interchanges q and q'; Q and Q' in the previous two paragraphs (see the lower part of Figure 3).

Figure 3: *From* c_N *to* c''_N *by Two Different Double-Arbitrages*

$$c \begin{array}{l} \xrightarrow{\text{agent } k \notin Q \text{ arbitrages}} (c'_Q, c_{-Q}) \xrightarrow{\text{agent } q \text{ arbitrages}} \\ \xrightarrow[\text{agent } k \notin Q' \text{ arbitrages}]{} (c'_{Q'}, c_{-Q'}) \xrightarrow[\text{agent } q' \text{ arbitrages}]{} \end{array} c''$$

It follows from the same argument that

(16) $$f_{q'}(c''; A) = \sum_{j \in Q'} f_j(c; A) .$$

Finally, Equation (15), symmetry, and Equation (16) imply that

(17) $$\sum_{i \in Q} f_i(c; A) = f_q(c''; A) = f_{q'}(c''; A) = \sum_{j \in Q'} f_j(c; A) .$$

The implied property of group symmetry turns out to be a normative basis of proportional rule. We show that the proportional rule can be characterized by group symmetry when two regularity conditions, *replication invariance* and *continuity*, are imposed. (Both are, of course, also implied by no-arbitrage.)

Replication invariance: For any bankruptcy problem $(c;A)$, if $(c';A')$ is an k-replica of $(c;A)$, i.e. $c' = (\underbrace{c,\ldots,c}_{k})$ and $A' = kA$, then $f(c';A') = (\underbrace{f(c;A),\ldots,f(c;A)}_{k})$, where k is a positive integer.

Replication invariance says that when a problem is replicated k times, the allocation of the larger problem is the k-replica of the original allocation.

(c)-Continuity: For any sequence of profiles $\{c^k\}$, if $c^k \to c$, then $f(c^k;A) \to f(c;A)$.

Theorem 2: *Proportional rule is the only rule that satisfies group symmetry, replication invariance and continuity.*[12]

Proof: Our plan is to show that if f satisfies the three axioms, then $\frac{f_i(c;A)}{f_j(c;A)} = \frac{c_i}{c_j}$ for any two agents i,j in a bankruptcy problem $(c;A)$. The case in which the ratio $\frac{c_i}{c_j}$ is rational number is considered first. We can identify two positive integers, k_i and k_j, such that $\frac{k_i}{k_j} = \frac{c_i}{c_j}$. The problem $(c;A)$ is replicated $\max\{k_i,k_j\}$ times. Group symmetry implies that all agents with c_i in the larger problem receive the same amount, say x_i, and all agents with c_j in the same problem receive the amount x_j. Replication invariance implies that

(18) $$f_i(c;A) = x_i \text{ and } f_j(c;A) = x_j.$$

Note that $\underbrace{c_i + \ldots + c_i}_{k_j} = \underbrace{c_j + \ldots + c_j}_{k_i}$ (in the larger problem). Group symmetry implies that $\underbrace{x_i + \ldots + x_i}_{k_j} = \underbrace{x_j + \ldots + x_j}_{k_i}$. Therefore, $\frac{f_i(c;A)}{f_j(c;A)} = \frac{x_i}{x_j} = \frac{k_i}{k_j}$.

The case in which $\frac{c_i}{c_j}$ is an irrational number can be dealt with by continuity.

Theorem 2 also covers the two-agent case. Moreover, it is tight. It can be verified that the following rule satisfies group symmetry and replication invariance, but not continuity.

(19) $$f(c;A)=\begin{cases}\text{uniform rule} & \text{if } c \text{ is an } k\text{ - replica of } (1,\pi) \text{ for any positive integer } k\\ \text{proportional rule} & \text{otherwise}\end{cases}$$

A rule that satisfies group symmetry and continuity, but not replication invariance can also be constructed, e.g.

(20) $$f(c;A)=\begin{cases}\text{proportional rule} & \text{if } 2\max\{c_i\}\le\sum c_i\\ \text{rule in Figure 4} & \text{otherwise}\end{cases}$$

[Insert Figure 4 here]

4. Conclusions

This paper proposes a new market-based approach to study the bankruptcy problem. It is shown that the no-arbitrage condition is a necessary and sufficient condition for the existence of equilibrium in the claims market. A striking result is that the equilibrium allocation is identical with the allocation according to the proportional rule.

The existing axiomatic literature on the bankruptcy problem has characterized the proportional rule based on strategic interpretations of the no-arbitrage condition, such as no advantageous reallocation and non-manipulability by merging and splitting. We show that the no-arbitrage condition can be used to simplify these characterizations of the proportional rule by replacing anonymity and continuity with more elementary properties.

No-arbitrage is also useful in normative analysis. A new normative axiom, group-symmetry, is derived from it. Group-symmetry, along with two regularity conditions, is used to provide a new normative characterization of the proportional rule.

[12] The same result has also been proved independently by Chambers and Thomson (2000).

Thus, the no-arbitrage condition is used to provide a common foundation for the proportional rule in a market framework, in a (axiomatic) strategic framework and also in a (axiomatic) normative framework. It is hoped that our analysis can enhance the understanding of the proportional rule, its central role and other related properties in the bankruptcy problem.

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Figure 4: *The Rule in Equation (20)*