ON THE EULER EQUATIONS OF HYPERBOLIC-DISCOUNTING GROWTH MODELS

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Abstract

Stimulated by recent studies on hyperbolic discounting, Barro (1999) examines the neoclassical model of capital accumulation using a continuous-time model and a general time preference function. However, the analysis seems rather complicated. This paper provides a complementary analysis to Barro (1999) by emphasizing the transparent derivation of the hyperbolic Euler equation and its intuition. To achieve these objectives, a discrete-time model and a specific time preference function first appearing in Phelps and Pollak (1968) are used. The analysis suggests that the derivation of the Euler equations in hyperbolic-discounting growth models shows similarity with the standard exponential-discounting case. Moreover, it shows that an interpretation of the hyperbolic Euler equation for the intertemporal consumption model in Harris and Laibson (2001) contains model-specific elements and therefore may not be valid for other dynamic models. This paper also extends some results in Phelps and Pollak (1968) and Long and Plosser (1983) by obtaining closed-form solutions for some hyperbolic-discounting growth models.

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1. INTRODUCTION

A standard assumption in dynamic economic analysis is the use of an *exponential discount function* (i.e., a constant rate of time preference). Important examples include the neoclassical model of capital accumulation and the intertemporal consumption model. A consequence of this assumption is the time-consistency property: the decision maker's optimal plan for future behavior chosen at a given date will be followed subsequently when he/she reconsiders the plan.

On the other hand, a *non-exponential discount function* (i.e., a time-varying rate of preference) causes a change in the relative valuation of utility level at different dates as the decision date evolves. This induces a time-inconsistent plan (Strotz, 1956). When the decision maker faces such an intertemporal struggle, the actual outcome will usually depend on whether or not this conflict is recognized by the decision maker, and whether or not commitment possibilities are available.

Based on Strotz's theoretical work, as well as empirical evidence in economics and psychology (such as Thaler, 1981; Ainslie, 1992), Laibson (1997) argues forcefully that individuals' discount functions are hyperbolic (i.e., the discount rate is higher in the near future than in the distant future) rather than exponential. For example, consumers are usually highly impatient about consumption between the current and next periods, but are more patient about choices between two far-off periods. Laibson and his collaborators (Laibson, 1997; Laibson *et al.*, 1998) examine several implications of hyperbolic discounting, such as the undersaving issues and the use of illiquid assets as a form of commitment.

Stimulated by these insights, Barro (1999) investigates whether the neoclassical model of capital accumulation would remain as the workhorse model for dynamic macroeconomics in the presence of hyperbolic discounting. His analysis suggests a basically positive answer. Moreover, his work yields testable relationships between the extent of commitment ability and the saving rate.

As in Barro (1999), this paper analyzes infinite-horizon one-sector growth models with hyperbolic agents. In fact, the paper is mainly motivated by Barro's analysis.[1] Barro (1999) examines the Ramsey model of capital accumulation in which the representative agent has a general time-varying discount function. He also uses a continuous-time framework. While the general specification of time preference (Eq. (2) in that paper) allows him to obtain results with different degrees of generality, the derivation seems rather long and complicated. For example, six pages are devoted to the analysis of the simplest case with log utility function. Moreover, perhaps because of the difficulty in obtaining sharp results in the general case, he discusses in a number of occasions the results based on a special hyperbolic discount function similar to that in Phelps and Pollak (1968) and Laibson (1997).

In order to obtain the solution in a more transparent way and to better understand its properties, this paper chooses a different strategy: using a discrete-time framework and starting with the tractable hyperbolic discount function suggested in Phelps and Pollak (1968). By modifying the standard dynamic programming technique to deal with the extra complication arising from hyperbolic discounting, it turns out that the derivation of the intertemporal Euler equation in this case shares much similarity with that in the standard exponential-discounting case. An interpretation is then

[1] The analysis of this paper is also motivated by Maskin and Tirole (1988) and, especially, Harris and Laibson (2001), which considers an intertemporal consumption model with stochastic but exogenous income, borrowing constraints and a constant interest rate. Each of the above papers obtains the solution by defining two value functions (corresponding to the structure of the problem). In particular, this paper bears close resemblance with Harris and Laibson (2001) in terms of the derivation of the hyperbolic Euler equation. However, there are differences in the hyperbolic Euler equations of the intertemporal consumption model (with exogenous interest rate and income processes) and the growth model (with interest rate and income processes endogenously determined); see the last paragraph of Section 3.

provided for the Euler equation of the class of hyperbolic-discounting growth models.

After deriving and interpreting the Euler equations for growth models with hyperbolic agents, this paper illustrates the usefulness of the Euler equation by obtaining closed-form solutions for some growth models. For this purpose, the paper further assumes a constant-relative-risk-aversion (CRRA) utility function and a Cobb-Douglas production function. Closed-form solutions are obtained in two special cases: the neoclassical growth model (with log utility function and complete depreciation of capital in one period) and the endogenous growth model of the Romer-Rebelo type (Romer, 1986; Rebelo, 1991). The analysis shows that some results in Phelps and Pollak (1968) and Long and Plosser (1983) can be generalized.

The rest of this paper is organized as follows. Section 2 derives the Euler equations for infinite-horizon growth models with hyperbolic discounting. Specifically, the agent maximizes an intertemporal additively-separable utility function, with a particular form of hyperbolic discount function first used in Phelps and Pollak (1968). It is assumed that the agent recognizes the intertemporal consequences of the hyperbolic preferences but is unable to commit future decision (except through affecting the value of the state variable). The analysis in Section 2 makes clear how the Euler equation can be obtained in a systematic manner, similar to the familiar case with exponential discounting. Section 3 provides an interpretation of the hyperbolic Euler equation. Sections 4 and 5 aim to obtain closed-form solutions, with Section 4 focusing on the neoclassical model of capital accumulation and Section 5 focusing on the endogenous growth models. Section 6 concludes.

2. DERIVATION OF THE HYPERBOLIC EULER EQUATION

This section considers growth models with hyperbolic agents. Specifically, it derives the Euler equations for infinite-horizon growth models when the hyperbolic agents recognize the time-inconsistent problem. (In Strotz's (1956) terminology, this is the case of 'consistent planning'; in the terminology of O'Donoghue and Rabin (1999), the hyperbolic agents are 'sophisticated', not 'naive'.) The analysis serves as a basis for (a) comparing the hyperbolic-discounting growth models with standard exponential-discounting growth models, and (b) deriving the results for some special cases of hyperbolic-discounting growth models in later sections.

In order to compare with existing results in the literature and to highlight the intuition of the hyperbolic Euler equation, the following analysis focuses on a class of one-sector growth models with fairly standard assumptions. A representative agent (or dynasty) framework is used, and the representative agent with hyperbolic discounting is assumed to maximize the following lifetime utility function:[2]

$$U\left(c_t\right) + \theta \sum_{j=1}^{\infty} \beta^j U\left(c_{t+j}\right), \tag{1}$$

where $0 < \theta \leq 1$, $0 < \beta < 1$, subscript t indexes time, c is consumption, and the instantaneous utility function is assumed to possess $U' > 0$, and $U'' < 0$. Parameter β is the long-run discount factor (with higher β meaning greater patience), and parameter θ is the extra factor for short-run discounting on top of β. The agent has the standard exponential discounting (or 'perfect altruism' according to Phelps and Pollak (1968), who analyze intergenerational altruism issues) if $\theta = 1$, and has hyperbolic discounting (or 'imperfect altruism' according to Phelps and Pollak (1968)) if $\theta < 1$. When θ is smaller than 1, the degree of impatience about consumption between the current and next periods is higher than the degree of impatience between any two adjacent periods in the future. The term $1 - \theta$ can be conveniently interpreted as the degree of hyperbolic discounting.

[2]The discount function $\left(1, \theta\beta, \theta\beta^2, ..., \theta\beta^j, ...\right)$ in the lifetime utility function (1) is first used in Phelps and Pollak (1968). Laibson (1997) and Harris and Laibson (2001) call it 'quasi-hyperbolic discount function' and contrast it with the class of 'generalized hyperbolic discount function' used in the psychology literature; see, for example, Figure I of Laibson (1997). For convenience, this discount function is simply called hyperbolic discount function in the following analysis.

The law of motion of this economy (which corresponds to the intertemporal resource constraint) is given by:

$$k_{t+j+1} = (1 - \delta)\, k_{t+j} + f\,(k_{t+j}) - c_{t+j} \equiv m\,(k_{t+j}) - c_{t+j}, \tag{2}$$

where $0 \square \delta \square 1$, k is capital stock (possibly in per capita term), $f\,(k)$ is the production function (usually in intensive form) relating gross output to capital, and the function $m\,(k)$ has the interpretation of undepreciated capital plus current output.

The above specification is essentially a standard one-sector growth model with the usual exponential discounting assumption replaced by the possibility of hyperbolic discounting. As mentioned before, this paper considers the behavior of sophisticated hyperbolic agents. Following Laibson (1997) and Barro (1999), the equilibrium of this economy can be interpreted as the outcome of a game played by a series of autonomous temporal 'selves'.

The following analysis focuses on Markov strategies, i.e., the choice variable is a function of the state variable(s). For the above problem, the state variable is capital stock and therefore, the Markov strategy is $c\,(k)$, consumption as a function of current capital stock. The Markov perfect equilibrium of this economy is defined as: a strategy sequence $\{c\,(k_j)\,, \forall j\}$ constitutes a Markov perfect equilibrium if $c\,(k_t)$ maximizes self t's intertemporal payoff given the equilibrium choices of other selves.

The problem can be solved by applying the idea of dynamic programming appropriately modified to deal with the complication caused by hyperbolic discounting. Given the structure of the problem, it is useful to define two value functions for each self, one at the current period (when he/she makes the decision) and another at any subsequent period. Without loss of generality, consider self t. Define $W\,(k_t)$ as the (maximized level of) intertemporal payoff at the beginning of period t, from the perspective of self t, given that the current state variable is k_t and all future selves will behave optimally. Also, define $F\,(k_{t+j})\,,\ j \geq 1$, as the future intertemporal payoff at the beginning of period $t+j$ $(j \geq 1)$, from the perspective of self t and evaluated at period $t+j$, given that the state variable at that period is k_{t+j} and all future selves will behave optimally. To distinguish between the two value functions, $F(k_{t+j})$ is called the continuation-value function and $W(k_t)$ the current-value function.

The value functions are related as follows. For the current period t,

$$\begin{aligned} W(k_t) &= \max_{h(k_t)} \left[U(h(k_t)) + \theta\beta F(m(k_t) - h(k_t)) \right] \\ &= U(c(k_t)) + \theta\beta F(m(k_t) - c(k_t)), \end{aligned} \tag{3}$$

where, assuming the presence of an interior solution, the optimal consumption function $c(k_t)$ is characterized by:[3]

$$U'(c(k_t)) = \theta\beta F'(m(k_t) - c(k_t)). \tag{4}$$

Also, for any future period $(j \geq 1)$, the continuation-value functions are related according to:[4]

$$F(k_{t+j}) = U(c(k_{t+j})) + \beta F(m(k_{t+j}) - c(k_{t+j})). \tag{5}$$

The next step is to combine the first-order condition and the marginal value of capital condition, similar to the analysis of, for example, Benveniste and Scheinkman

[3] One can generalize the analysis to allow for the possibility of corner solution (as in Harris and Laibson, 2001). In that case, the equality sign in (3) should be replaced by an inequality sign. Since an interior solution is present in many growth models (such as those in Sections 4 and 5), there is no loss of generality in focusing only on interior solutions for these classes of growth models.

[4] Note that Eq. (5) links two adjacent continuation-value functions, while Eq. (3) links the current-value function with the continuation-value function at the next period. This can be seen from the fact that the discount factor between any two future adjacent periods in Eq. (5) is β, while the discount factor between the current and next periods in Eq. (3) is $\theta\beta$. Note also that the consumption function $c(k_t)$ in (3) is chosen by the current self, while $c(k_{t+j})$ in (5) is chosen by another self, self $t+j$. (However, by exploiting the stationarity of the infinite-horizon problem, the optimal choice of self $t+j$ is also characterized by a first-order condition similar to (4)).

(1979) for the exponential-discounting case. For the growth model with hyperbolic discounting, the marginal value of capital is obtained as follows. Differentiating the value function $W(k_t)$ with respect to the state variable k_t gives:

$$\begin{aligned} W'(k_t) &= U'(c(k_t))\,c'(k_t) + \theta\beta F'(m(k_t) - c(k_t))\left[m'(k_t) - c'(k_t)\right] \\ &= U'(c(k_t))\,m'(k_t), \end{aligned} \tag{6}$$

where the second equality makes use of the first-order condition (4). The function $m'(k_t) \equiv \frac{\partial m(k_t)}{\partial k_t}$ can be interpreted as a one-period (endogenous) gross real interest rate.

The final step in obtaining the Euler equation is to combine the above conditions. With exponential discounting (and therefore, $W(k) = F(k)$ for an infinite-horizon problem), the usual procedure is to immediately combine the first-order condition and the marginal value of capital condition. However, with hyperbolic discounting, $F'(k)$ appears in (4) and $W'(k)$ appears in (6), and these two functions are different. The remaining analysis makes use of the relationship between the current-value function and the continuation-value function. Combining self $t+1$'s current-value function (i.e., $W(k_{t+1})$ with state variable k_{t+1}) and self t's continuation-value function for period $t+1$ (i.e., $F(k_{t+1})$ with the same state variable k_{t+1}) gives:

$$\begin{aligned} W(k_{t+1}) &= U(c(k_{t+1})) + \theta\beta F(m(k_{t+1}) - c(k_{t+1})) \\ &= U(c(k_{t+1})) + \theta\left[F(k_{t+1}) - U(c(k_{t+1}))\right] \\ &= (1-\theta)\,U(c(k_{t+1})) + \theta F(k_{t+1}). \end{aligned} \tag{7}$$

Eq. (7) suggests that the current-value function of self $t+1$ is related to the continuation-value function of self t at period $t+1$ in a particular way: $W(k_{t+1})$ is a weighted average of $F(k_{t+1})$ and the current-period (maximized) utility level.[5]

[5] An alternative (and perhaps more intuitive) way to look at (7) is as follows:

$$W(k_{t+1}) = U_{t+1} + \theta\beta U_{t+2} + \theta\beta^2 U_{t+3} + ...$$

Differentiating (7) with respect to k_{t+1} gives:

$$\theta F'(k_{t+1}) = W'(k_{t+1}) - (1-\theta) U'(c(k_{t+1})) c'(k_{t+1}), \tag{7a}$$

where $c'(k)$ is the marginal propensity to consume (with respect to capital).

Combining (4), (6) and (7a) gives the intertemporal Euler equation for the growth model with hyperbolic discounting:

$$U'(c(k_t)) = \theta\beta F'(k_{t+1})$$

$$= \beta [W'(k_{t+1}) - (1-\theta) U'(c(k_{t+1})) c'(k_{t+1})]$$

$$= \beta [m'(k_{t+1}) - (1-\theta) c'(k_{t+1})] U'(c(k_{t+1})). \tag{8}$$

Note that when $\theta = 1$, (8) becomes the standard Euler equation (for exponential discounting):

$$U'(c(k_t)) = \beta m'(k_{t+1}) U'(c(k_{t+1})). \tag{8a}$$

3. INTERPRETATION OF THE HYPERBOLIC EULER EQUATION

For the exponential-discounting case, a standard interpretation of the Euler equation (8a) comes from the perturbation arguments. Starting from the equilibrium path, assume that consumption is lowered at period t (by Δc_t, an infinitesimal amount) and increased at period $t+1$ (by $m'(k_{t+1}) \Delta c_t$, the product of Δc_t and the gross interest

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and

$$F(k_{t+1}) = U_{t+1} + \beta U_{t+2} + \beta^2 U_{t+3} + ...$$

where $U_{t+j} \equiv U(c(k_{t+j}))$ represents the utility level at period $t+j$ along the equilibrium path. Combining these two equations yields

$$\theta [F(k_{t+1}) - U_{t+1}] = W(k_{t+1}) - U_{t+1},$$

which is the same as Eq. (7).

rate), while holding consumption at other periods constant. The utility level at period t decreases by $U'(c_t)\Delta c_t$, but the future utility level (evaluated at period $t+1$) increases by $m'(k_{t+1})U'(c_{t+1})\Delta c_t$. Evaluating both terms at period t, one obtains the Euler equation (8a) characterizing the equilibrium consumption path.

The above perturbation arguments, however, do not go through in the presence of hyperbolic discounting, because the sequence of decisions is made by different selves and the current self cannot commit to a particular consumption path in the future. Instead, the hyperbolic agent (at a particular period) will figure out how his/her choice of current consumption would affect the stock of capital (the state variable) and how this change in capital would affect the consumption choices of future selves. When consumption at period t is decreased by Δc_t (from the level of the equilibrium path), the utility level at period t decreases by $U'(c_t)\Delta c_t$, but the future utility level (evaluated at period t) increases by $\theta\beta F'(k_{t+1})\Delta c_t$.[6] Combining with the marginal value of capital condition (6) and the fact that $F(k_{t+1})$ is a linear combination of $W(k_{t+1})$ and the current-period (maximized) utility level $U(c_{t+1})$, one has the Euler equation (8).

While the interpretation of Euler equation (8) under hyperbolic discounting is not as straightforward and intuitive as (8a) for exponential discounting, some discussion can still be made by looking at the ratio of marginal utility of consumption at the current period to that at the next period $(= U'(c_t)/U'(c_{t+1}))$. For exponential discounting, (8a) shows that the marginal utility ratio equals to the product of the (constant) discount factor β and the gross interest rate.[7] With hyperbolic discounting, Eq. (8)

[6] Similar argument can be made for the case of exponential discounting. When consumption at period t is decreased by Δc_t (from the level of the equilibrium path), the utility level at period t decreases by $U'(c_t)\Delta c_t$, but the future utility level (evaluated at period t) increases by $\beta W'(k_{t+1})\Delta c_t$. Using (6), one obtains (8a).

[7] For the exponential-discounting case, this condition is equivalent to the more familiar condition that the marginal rate of substitution $(= U'(c_t)/\beta U'(c_{t+1}))$ equal to the gross interest rate (i.e., the

shows that the marginal utility ratio further depends on two factors: $(1-\theta)$ and $c'(k_{t+1})$.

The intuition of this dependence is that in the presence of hyperbolic discounting, the current self values marginal saving more than marginal consumption at any future period, since he/she expects future selves to over-consume relative to the consumption path that he/she prefers those future selves to choose. Since $1-\theta$ measures the degree of hyperbolic discounting and thus indicates the magnitude of the difference between future marginal saving and marginal consumption, it matters in the current self's decision. On the other hand, $c'(k_{t+1})$ matters because it determines how a marginal unit of capital at period $t+1$ will be divided between saving and consumption. When either $1-\theta$ (i.e., the degree of hyperbolic discounting) or $c'(k_{t+1})$ (i.e., marginal propensity to consume in the next period) is higher, the marginal utility ratio is lower, and thus, current consumption is higher (because of diminishing marginal utility of consumption).

An interesting observation about the hyperbolic Euler equation (8) for an infinite-horizon growth model is that it differs, slightly, from its counterpart for an infinite-horizon buffer stock consumption model with constant interest rate and exogenous income, given by Eq. (8) of Harris and Laibson (2001). In particular, Harris and Laibson (2001, p. 936) suggest that for the intertemporal consumption model with hyperbolic discounting, the endogenous 'effective discount factor' is a weighted average of the short-run discount factor $(\theta\beta)$ and the long-run discount factor (β). A comparison of Eq. (8) of Harris and Laibson (2001) and Eq. (8) of this paper sug-

relative price of current versus next period's consumption). Note that for the hyperbolic-discounting case, the marginal rate of substitution between the current and next periods $(= U'(c_t)/\theta\beta U'(c_{t+1}))$ is not the same as that between any two future adjacent periods $(= U'(c_{t+j})/\beta U'(c_{t+j+1}),\, j \geq 1)$. As a result, it is clearer to base the discussion of the Euler equation (8) on the marginal utility ratio than on the marginal rate of substitution.

gests that their interpretation does not extend to the classes of growth models. While there are similarities in the derivation of the hyperbolic Euler equations for these two models, the interpretation of the respective Euler equations contains model-specific elements.

4. THE NEOCLASSICAL MODEL OF CAPITAL ACCUMULATION

In the remaining sections of this paper, the hyperbolic Euler equation (8) is applied to some growth models. From now on, this paper assumes a Cobb-Douglas production function and a CRRA utility function:

$$U\left(c_{t+j}\right)=\frac{c_{t+j}^{1-\rho}-1}{1-\rho}, \tag{9}$$

where ρ ($\rho>0$) is the relative risk aversion coefficient. These are standard assumptions in the literature; moreover, they are required for the phenomenon of balanced growth (King *et al.*, 1988). Note that when ρ tends to 1, $U(c)$ in (9) tends to $\ln c$.

This section considers the basic neoclassical model of capital accumulation with a constant level of technology and labor input.[8] The intertemporal utility function is given by (1) with instantaneous utility function (9). On the other hand, the production technology is represented by a standard Cobb-Douglas production function:

$$y_t=Ak_t^{\alpha}N^{1-\alpha}=Ak_t^{\alpha}, \tag{10}$$

where $0<\alpha<1$; the (inelastically supplied) labor input at each period, N, has been normalized to be 1; and y_t and k_t are respectively the output and the associated capital input at period t. (As labor input has been normalized to 1, y_t and k_t can also be interpreted respectively as output per worker and capital per worker.) Combining

[8]It is straightforward to generalize the model to incorporate constant and positive growth rate of exogenously specified technology and population. To save space and notation, this paper focuses on the basic model. The same strategy is followed in Barro (1999, Section III).

(10) with the intertemporal resource constraint, the law of motion (2) becomes:

$$k_{t+j+1} \equiv m\left(k_{t+j}\right) - c\left(k_{t+j}\right) = (1-\delta)\, k_{t+j} + Ak_{t+j}^{\alpha} - c\left(k_{t+j}\right). \tag{11}$$

With CRRA utility function (9) and law of motion (11), the Euler equation (8) becomes:

$$\left[\frac{c\left(k_{t+1}\right)}{c\left(k_t\right)}\right]^{\rho} = \beta\left[1 - \delta + \alpha A k_{t+1}^{\alpha-1} - (1-\theta)\, c'\left(k_{t+1}\right)\right]. \tag{12}$$

For this economy, a closed-form solution does not exist in general, except for the special case of $\rho = 1$ (log utility) and $\delta = 1$ (complete depreciation of capital in one period).[9] With these two restrictions, the Euler equation (12) becomes:

$$\frac{c\left(k_{t+1}\right)}{c\left(k_t\right)} = \beta\left[\alpha A k_{t+1}^{\alpha-1} - (1-\theta)\, c'\left(k_{t+1}\right)\right]. \tag{12a}$$

In Appendix 1, it is shown that the hyperbolic Euler equation (12a) for the neoclassical model with $\rho = 1$ and $\delta = 1$ is satisfied when consumption is a time-invariant fraction of output:

$$c\left(k_t\right) = \left[\frac{1-\beta\alpha}{1-\beta\alpha(1-\theta)}\right] Ak_t^{\alpha}. \tag{13}$$

5. WITH ENDOGENOUS GROWTH

This section provides an analysis for endogenous growth models with hyperbolic agents. Phelps and Pollak (1968, pp. 186-187) conjecture that the endogenous saving rate of the economy is constant under the following assumptions: (a) a CRRA utility function, (b) a constant marginal productivity of capital, and (c) no depreciation in capital. The analysis of this section extends the above results of Phelps and Pollak by showing that for the endogenous growth models of the Romer-Rebelo type (Romer, 1986; Rebelo, 1991) with CRRA utility function, a closed-form solution (with constant

[9] With exponential discounting, it is well-known (from, for example, Long and Plosser, 1983) that a closed-form solution exists under these conditions. The analysis of this section shows that this result can be extended to hyperbolic discounting.

saving rates) exists. Moreover, it makes clear that the assumption of no depreciation of capital is not required.

In this section, the Romer (1986) model with Cobb-Douglas production function, together with the CRRA utility function (9), will be used. This model incorporates the AK model (Rebelo, 1991) as a special case, which is essentially the model considered in Phelps and Pollak (1968).

The production function (of a Cobb-Douglas form) used in Romer (1986) is:

$$y_t \equiv f\left(k_t; \overline{k}_t\right) = A k_t^{\alpha} \overline{k}_t^{1-\alpha}. \tag{14}$$

where $0 < \alpha < 1$, k_t is the capital stock of an individual agent and $\overline{k}_t$ is the average capital stock of the economy. As a result, the law of motion of this economy is described by:

$$k_{t+j+1} \equiv m\left(k_{t+j}; \overline{k}_{t+j}\right) - c\left(k_{t+j}\right) = (1-\delta)\, k_{t+j} + A k_{t+j}^{\alpha} \overline{k}_{t+j}^{1-\alpha} - c\left(k_{t+j}\right). \tag{15}$$

With Eqs. (9) and (15), the Euler equation (8) becomes:

$$\left[\frac{c\left(k_{t+1}\right)}{c\left(k_t\right)}\right]^{\rho} = \beta m'\left(k_{t+1}; \overline{k}_{t+1}\right) - \beta\left(1-\theta\right) c'\left(k_{t+1}\right), \tag{16}$$

where the gross interest rate is given by:

$$m'\left(k_t; \overline{k}_t\right) \equiv \frac{\partial m\left(k_t; \overline{k}_t\right)}{\partial k_t} = 1 - \delta + A\alpha k_t^{\alpha-1} \overline{k}_t^{1-\alpha}. \tag{17}$$

At the equilibrium path of this (homogenous-agent) economy, the following condition holds:

$$k_t = \overline{k}_t. \tag{18}$$

In Appendix 2, it is shown that the variables such as the equilibrium level of capital will evolve according to:

$$k_{t+1} \equiv m\left(k_t; k_t\right) - c\left(k_t\right) = \left(1+\gamma\right) k_t, \tag{19}$$

where γ is the (net) growth rate to be determined endogenously. Moreover, it is shown that Euler equation (16) is satisfied when consumption is a constant fraction of capital (or equivalently, a constant fraction of output given that the production function (14) is of constant returns to scale with respect to accumulable inputs):

$$c\left(k_t\right) = dk_t, \tag{20}$$

where d $(0 < a < 1)$ is a constant to be determined.

Finally, it is shown in Appendix 2 that coefficients γ and d (which are defined in (19) and (20) respectively) are determined jointly by:

$$1 + \gamma = 1 - \delta + A - d, \tag{21}$$

and

$$(1 - \delta + A - d)^{\rho} = \beta \left[1 - \delta + A\alpha - (1 - \theta)\, d\right]. \tag{22}$$

It can be observed from (21) and (22) that the values of γ and d depend on the exogenous parameters θ, β, ρ, α, δ and A. Moreover, when $\rho = 1$ (log utility), it is straightforward to show that the optimal consumption function is given by:

$$c\left(k_t\right) = dk_t = \left[\frac{1 - \delta + A - \beta\left(1 - \delta + A\alpha\right)}{1 - \beta\left(1 - \theta\right)}\right] k_t. \tag{20a}$$

6. CONCLUSION

Since Strotz's (1956) seminal work, the time-inconsistency problem has become well-known. However, not much work has been done on this topic, except a few recent important studies examining the relevance and implications of hyperbolic discounting. Given the importance of hyperbolic discounting, this paper provides an analysis of growth models with hyperbolic agents. It is motivated in particular by the analysis of Barro (1999) who uses a continuous-time model with a general time preference function. Unlike Barro, this paper uses a discrete-time model and the specific discount function suggested in Phelps and Pollak (1968) and Laibson (1997). Within this framework, the Euler equations of infinite-horizon growth models under hyperbolic discounting are derived and an interpretation is given.

Compared with Barro's (1999) analysis, this paper emphasizes the transparency of the derivation of the Euler equation and the intuition of the equation. Moreover, it shows that a particular interpretation in Harris and Laibson (2001) regarding the hyperbolic Euler equation (that the endogenous effective discount factor is a weighted average of the short-run discount factor and the long-run discount factor) is specific to the intertemporal consumption model (with exogenous income and constant interest rate) and therefore is not necessarily valid for other intertemporal models.

In the derivation of the hyperbolic Euler equation in this paper, there are elements of generality as well as elements of speciality. The general specification of $m(k)$ in Eq. (2) includes many growth models (such as the neoclassical growth model discussed in Section 4, and some endogenous growth models discussed in Section 5). On the other hand, only one-sector growth models are considered and one state variable k is assumed in Eq. (2). This is chosen so that the Euler equation can be easily derived and interpreted. The procedure can be extended to more sophisticated situation. In fact, the Euler equation for an endogenous growth model with externality (Romer, 1986) has been considered in Section 5. Similar analysis can also be conducted for hyperbolic-discounting growth models with more state variables.

This paper also provides closed-form solutions for some special cases of one-sector growth models with hyperbolic discounting. The two special cases considered in this paper are: the neoclassical growth model with log utility function and 100% depreciation of capital in one period, and the Romer-Rebelo endogenous growth model. In both cases, it turns out that the endogenous saving rate is time-invariant. The analysis in this paper shows that the results for the neoclassical growth model (Long and Plosser, 1983) can be extended to hyperbolic discounting. Moreover, it makes clear the reasons behind a conjecture suggested in Phelps and Pollak (1968) and shows that some of their results can be extended to the Romer-Rebelo endogenous growth models.

The transparency of the derivation of the hyperbolic Euler equation suggests that similar analysis could be useful for future work on other growth models, such as those involving more realistic yet more complicated situation. For example, an analysis of the role of government spending on economic growth in the presence of hyperbolic discounting is currently analyzed in another project.

APPENDIX 1

Assume that consumption is a time-invariant fraction of output:

$$c\left(k_t\right) = bAk_t^{\alpha}, \tag{A1}$$

where b $(0 < b < 1)$ is a constant to be determined. From (A1), it follows that $c'\left(k_t\right) = b\alpha Ak_t^{\alpha-1}$. As a result, (12a) becomes:

$$\frac{bAk_{t+1}^{\alpha}}{bAk_t^{\alpha}} = \beta\left[\alpha Ak_{t+1}^{\alpha-1} - (1-\theta)\, b\alpha Ak_{t+1}^{\alpha-1}\right].$$

This leads to

$$k_{t+1} = \beta\left[1 - (1-\theta)\, b\right]\alpha Ak_t^{\alpha}. \tag{A2}$$

Also, substituting (A1) and $\delta = 1$ into (11) gives

$$k_{t+1} = Ak_t^{\alpha} - bAk_t^{\alpha} = (1-b)\, Ak_t^{\alpha}. \tag{A3}$$

Combining (A2) and (A3), parameter b is determined as:

$$b = \left[\frac{1-\beta\alpha}{1-\beta\alpha\,(1-\theta)}\right]. \tag{A4}$$

Substituting (A4) into (A1), the optimal consumption function for this special case of the neoclassical model of capital accumulation is given by (13) in the text.

APPENDIX 2

To see that a closed-form solution exists for the Romer-Rebelo model, note that the marginal product of capital for production function (14) at a given level of average capital is given by:

$$f'\left(k_t;\overline{k}_t\right) \equiv \frac{\partial f\left(k_t;\overline{k}_t\right)}{\partial k_t} = A\alpha k_t^{\alpha-1}\overline{k}_t^{1-\alpha}. \tag{A5}$$

With (18), the gross interest rate at equilibrium is a constant given by:

$$m'\left(k_t;k_t\right) = (1-\delta) + f'\left(k_t;k_t\right) = 1-\delta+A\alpha. \tag{17a}$$

Combining the result in (17a) with the idea in endogenous growth theory that the variables can have sustained growth endogenously in the presence of constant returns to scale to accumulable inputs (Rebelo, 1991), it is conjectured that the variables such as the equilibrium level of capital will evolve according to (19).

Since the first term on the right-hand side of the Euler equation (16) is a constant at equilibrium (with $k = \overline{k}$) according to (17a) and the equilibrium level of capital stock grows at a constant rate according to (19), it is conjectured that (16) is satisfied when consumption is given by (20).

The reason for the above conjecture is as follows. With consumption function (20), $c'\left(k_t\right) = d$ and thus, the second term on the right-hand side of (16) is constant. Combining (20) and a constant growth rate according to (19), the left-hand side of the Euler equation (16) is also constant. As a result, the conjecture of (19) and (20) is confirmed.

The remaining step is to find the undetermined coefficients γ in (19) and d in (20). Substituting (18), (19) and (20) into the law of motion (15) leads to (21). On the other hand, substituting (18), (19), (20), (21) and (17a) into the Euler equation (16) leads to (22). Thus, d and γ can be solved simultaneously from (21) and (22). Specifically, d is obtained from (22) and then γ is determined from (21).

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